Unit: Parametric Equations and Polar Coordinates **Module:** Polar Functions and Area

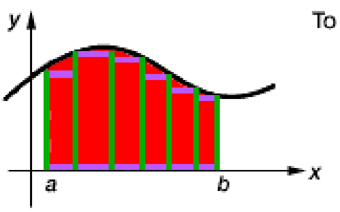
Heading Towards the Area of a Polar Region

key concepts:

- In Cartesian coordinates, the area under a curve is based on dividing the region into infinitely thin rectangles and summing their areas with a definite integral.
- In Polar coordinates, wedges are made infinitely thin and then summed to find the area under a curve.

Review of area under curves in Cartesian coordinates

Cartesian coordinates



To find the area, use the formula:

$$\text{Area} = \underbrace{\int_a^b}_{\text{the sum from } a \text{ to } b} \overbrace{f(x)}^{\text{height of rectangle}} \underbrace{dx}_{\text{base of rectangle}}$$

Q Can you write the analogous formula when working in polar coordinates?

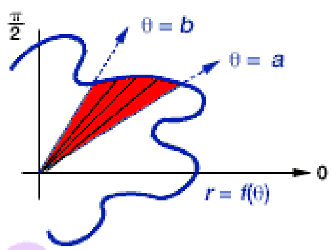
$$\cancel{\int_a^b f(\theta)\,d\theta}$$

In the Cartesian coordinate system, you can determine the area under a curve by partitioning the region into infinitely thin rectangles.

The height of each rectangle corresponds to the value of the function and the base of each rectangle is a tiny change in *x*.

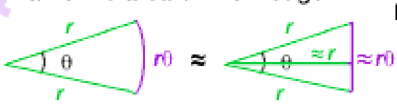
Unfortunately, the same formula doesn't work with polar coordinates.

The area of a wedge



Think of breaking the shape into wedges.

Q What is the area of the wedge?



Because the wedge is so narrow, its height is indistinguishable from the length of its sides.

The base of the triangle is roughly the same as the arc length of the wedge $r\theta$.

A As the wedge becomes very thin its area is $\frac{1}{2}r(r\theta) = \frac{1}{2}r^2\theta$.

Q What is the sum of all the thin wedges that make up the pie shape?

A $$\boxed{\text{Area} = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta}$$

For polar coordinates, you need to partition the region into wedges. The wedges are analogous to the rectangles of Cartesian areas.

The wedges are defined by a radius (r) and an angle (θ). By the arc length formula, the base of the wedge is $r\,\theta$. As the wedges get infinitely thin, they begin to resemble triangles whose height is r and whose base is $r\,\theta$.

Using these values, you can calculate the area of the wedge.

The radius of each wedge is equal to the value of the function at an angle theta between the limits of integration. In other words, r is replaced by $f(\theta)$. As the wedges get infinitely thin, the angle becomes a small change in θ, noted $d\theta$.

Calculus Lecture Notes

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Unit: Parametric Equations and Polar Coordinates **Module:** Polar Functions and Area

Finding the Area of a Polar Region – Part One

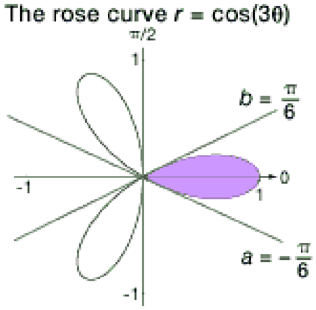
key concepts:

- When finding the area under a polar curve, you must choose limits of integration that sweep out the curve smoothly and cleanly.

Setting up the problem

GOAL: To find the area of a single petal of a rose curve

The rose curve $r = \cos(3\theta)$



$$\text{Area} = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$$

Oops, Don't forget $\boxed{\frac{1}{2}}\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$

Find the area of the petal centered on $\theta = 0$.

The formula for the area of a polar curve is:

$$\text{Area} = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$$

The a and b represent the smallest and largest values of θ that the petal sweeps out.

Remember:
Each ray can be represented by an infinite number of θ-values.

Be sure to choose values of a and b with $a < b$ such that a and b b smoothly and continuously sweep out the curve.

Here you want to find the area of the shaded petal of the rose curve given by $r = \cos(3\theta)$.

Use the formula for the area under a polar curve.

Be careful when determining the limits of integration. Since every angle has infinitely many coterminal angles, you must choose values that sweep out the area you desire. This should happen smoothly and continuously.

In most cases the first limit (a) should be less than the second limit (b).

If you want to have b equal $\pi/6$ in this example, then you should choose a equal to $-\pi/6$, not $11\pi/6$.

A helpful trigonometric identity

MAJOR ASIDE

$$\cos(2\varphi) = \cos^2\varphi - \sin^2\varphi$$ Double-angle formula

$$= \cos^2\varphi - (1-\cos^2\varphi)$$ From the Pythagorean theorem: $\sin^2\varphi + \cos^2\varphi = 1$, $\sin^2\varphi = 1 - \cos^2\varphi$

$$= 2\cos^2\varphi - 1$$

$$\cos(2\varphi) = 2\cos^2\varphi - 1$$

$$\cos(2\varphi) + 1 = 2\cos^2\varphi$$ Add one to both sides.

$$\boxed{\cos^2\varphi = \frac{\cos(2\varphi)+1}{2}}$$ Divide both sides by two.

Now you have a "COOL SUBSTITUTION" that allows you to replace a squared trigonometric function with a simple trigonometric function.

You are now ready to apply this "COOL SUBSTITUTION" to the integral and find the area of the rose petal.

$$\text{Area} = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$$

If the integral only had the first power of cosine, then that would be simple to integrate. But this integral has cosine squared, with no sine factor to allow you to use u-substitution.

Here is the development of an identity that will simplify the integral. Start with the double-angle identity for cosine.

Use the Pythagorean identity to eliminate the sine expression.

Solving for cosine squared produces an expression with just the first power of cosine.

In the next tutorial you will use this identity to substitute into the integral an expression that is easier to integrate.

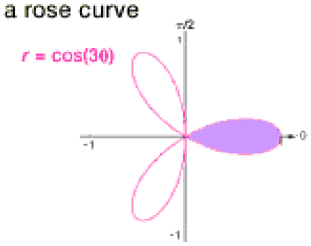
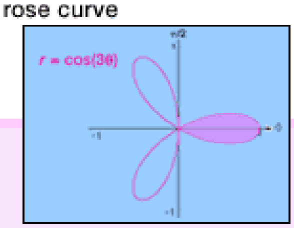
Unit: Parametric Equations and Polar Coordinates **Module:** Polar Functions and Area

Finding the Area of a Polar Region – Part Two

key concepts:

- Calculating the area under a polar curve involves using the finding the proper limits of integration and evaluating the following integral:

$$\frac{1}{2}\int_b^a [f(\theta)]^2\, d\theta\,.$$

Finding the area of a rose petal GOAL: Find the area of a petal of a rose curve centered at $\theta = 0$.  $\text{Area} = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$ $= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$ $= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{\cos(6\theta)+1}{2}\, d\theta$ Use the "cool substitution." Don't forget to replace the φ with a 3θ so that $2\varphi = 2(3\theta) = 6\theta$.	In the last lecture you were applying the formula for the area under a polar curve. To find the area of one petal of this rose curve, you first identified proper limits of integration and plugged in the expression for the function *f*. Then you developed a cool trig identity so that you didn't have to integrate a cosine squared term.
Finding the area of a rose petal GOAL: Find the area of a petal of a rose curve centered at $\theta = 0$.  Finding the antiderivative $\frac{1}{4}\int [\cos(6\theta)+1]\, d\theta$ $= \frac{1}{4}\int \cos(6\theta)\, d\theta + \frac{1}{4}\int 1\, d\theta$ The integral of a sum is the sum of the integrals. $= \frac{1}{4}\left(\frac{1}{6}\right)\sin(6\theta) + \frac{1}{4}\theta$ The *u du* substitution technique $\boxed{= \frac{1}{24}\sin(6\theta) + \frac{1}{4}\theta}$ $= \frac{1}{24}\sin(6\theta) + \frac{1}{4}\theta \Big\|_{-\pi/6}^{\pi/6}$ $= \frac{1}{24}\sin(\pi) + \frac{1}{4}\frac{\pi}{6} - \left[\frac{1}{24}\sin(-\pi) - \frac{1}{4}\left(-\frac{\pi}{6}\right)\right]$  $= \frac{\pi}{24} + \frac{\pi}{24} = \boxed{\frac{\pi}{12}}$ The area of one petal of $r = \cos(3\theta)$ is $\frac{\pi}{12}$, an elegant number for an elegant curve.	Forget about the limits of integration for a moment and focus on the antiderivative. Move the 1/2 out to make 1/4 the constant multiplier. Then split the integral into two parts. Be careful! Because the antiderivative of cos (6θ) requires a *u*-substitution, you need a factor of 1/6 in front of the sine term. Simplify to get a clean expression for the antiderivative. Now bring back the limits of integration. Evaluating the expression at the limits of integration produces an elegant result for the area of one petal of the rose curve.

Area of a Region Bounded by Two Polar Curves – Part One

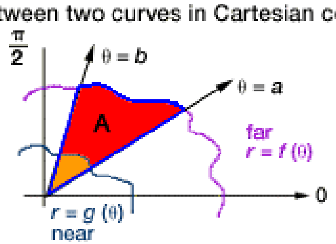
key concepts:

- The area of a region bound between two polar curves is
$$A=\frac{1}{2}\int_b^a\left\{[f(\theta)]^2-[g(\theta)]^2\right\}d\theta .$$
- The limits of integration occur where the curves intersect.

The formula for the area bound by two polar curves

GOAL: Find the area of a region bound by two polar curves.

Use the same principle for finding the area of a region between two curves in Cartesian coordinates.



Q: How do you find the area of the shaded region?

A: Subtract the area of the wedge bound by the inner curve from the area of the wedge bound by the outer curve.

The area bound by two polar curves is $A=\frac{1}{2}\int_a^b[f(\theta)^2-g(\theta)^2]d\theta$.

Or more intuitively, $A=\frac{1}{2}\int_a^b[\text{far}^2-\text{near}^2]d\theta$.

The same method for finding the area between two curves in Cartesian coordinates works for polar coordinates.

The idea is to find the area under the first curve and subtract from it the area under the second curve.

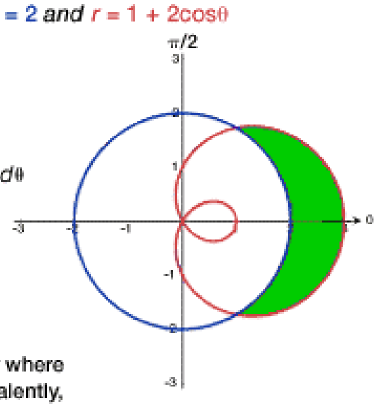
You can write the formula as a single integral that is based on the formula for finding the area under a polar curve.

Think of the process as calculating the are under the "far" curve and subtracting the area under the "near" curve.

Using the formula for the area bound by two polar curves

Example! The area bound by $r = 2$ and $r = 1 + 2\cos\theta$

Q: What is the area of the crescent shaped region between the two curves?



A: $A=\frac{1}{2}\int_a^b[\text{limaçon}^2-\text{circle}^2]d\theta$

Q: What are the limits of integration, a and b?

A: The limits of integration occur where the curves intersect, or equivalently, where their r-values coincide.

$$2=1+2\cos\theta$$
$$1=2\cos\theta$$
$$\frac{1}{2}=\cos\theta$$

Consider the shaded area bound by these two curves.

For this region, the "far" curve is the limaçon and the "near" curve is the circle.

You still need the limits of integration, a and b. These will be the θ-values where the two curves coincide.

Set the functions equal to each other and solve for θ.

You may need to recall how to solve trigonometric equations like this one.

Area of a Region Bounded by Two Polar Curves – Part One

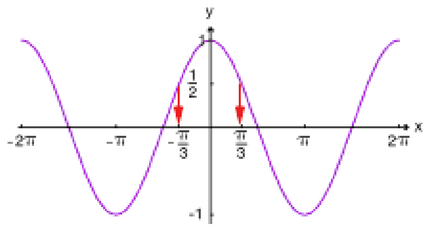
Using the formula for the area bound by two polar curves

Example! The area bound by $r = 2$ *and* $r = 1 + 2\cos\theta$

$$\frac{1}{2} = \cos\theta$$

Q: What are the values of θ that have a cosine of 1/2?

The cosine function



A: The first negative value of θ that has a cosine of 1/2 is $-\frac{\pi}{3}$.
The first positive θ-value that has a cosine of 1/2 is $\frac{\pi}{3}$.

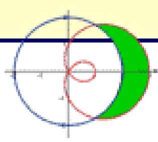
Solving this trigonometric equation is the same as finding the values of θ for which the cosine is 1/2.

It is always a good idea to draw a picture of what is going on.

Since the region is in the fourth and first quadrants, you want the first negative value of θ and the first positive value of θ that satisfy the equation. These values are $-\pi/3$ and $\pi/3$.

Setting up the integral

Example! The area bound by $r = 2$ *and* $r = 1 + 2\cos\theta$



$$A = \frac{1}{2}\int_a^b [\text{far}^2 - \text{near}^2]\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(1+2\cos\theta)^2 - 2^2\right]d\theta$$

The limaçon is the far curve and the circle is the near curve.

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(1+4\cos\theta+4\cos^2\theta) - 4\right]d\theta$$

FOIL

$$= \underbrace{\frac{1}{2}\int_{-\pi/3}^{\pi/3}[-3+4\cos\theta]\,d\theta}_{\text{easy}} + \underbrace{\frac{1}{2}\int_{-\pi/3}^{\pi/3}4\cos^2\theta\,d\theta}_{\text{challenging}}$$

Split the integral into easy terms and more difficult terms.

Hints for working out the challenging part of the integral:

1. "4" is just a constant, so you can pull it out of the integral.
2. "**Cool Substitution**". $\cos^2\varphi = \frac{\cos(2\varphi)+1}{2}$

Try this integration on your own...

Now you can plug all information into the formula.

You will need to expand the first binomial to make the integral easier to evaluate.

When you combine like terms, you will see some terms that are not difficult to integrate and others that are more challenging.

Remember, there is a cool substitution for transforming cosine squared into an expression that is easier to integrate. In the next tutorial you will see how to finish this example.

Unit: Parametric Equations and Polar Coordinates **Module:** Polar Functions and Area

Area of a Region Bounded by Two Polar Curves – Part Two

key concepts:

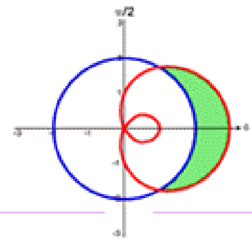
- When evaluating an integral whose integrand is a sum, break it into parts and integrate the easy terms separately from the challenging ones.

Calculating the area bound by two polar curves

Example! $r = 2$ and $r = 1+2\cos\theta$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(1+2\cos\theta)^2 - 2^2\right]d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}[-3+4\cos\theta]\,d\theta + \frac{1}{2}\int_{-\pi/3}^{\pi/3}4\cos^2\theta\,d\theta$$



The shaded area bounded by these two curves can be calculated using an integral.

It usually helps to break large problems into smaller parts. The parts are usually easier to handle individually.

The easy part of the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3}[-3+4\cos\theta\,]\,d\theta = \frac{1}{2}(-3\theta+4\sin\theta)\Big|_{-\pi/3}^{\pi/3}$$

Start with the part of the integral that does not require any substitutions. Remember that you are integrating with respect to θ.

$$= \frac{1}{2}\left[-\pi+4\sin\left(\frac{\pi}{3}\right)\right] - \frac{1}{2}\left[\pi+4\sin\left(-\frac{\pi}{3}\right)\right]$$

Use square brackets to remind yourself to distribute the minus sign.

$$= -\frac{\pi}{2}+\cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)-\frac{\pi}{2}-\cancel{2}\left(-\frac{\sqrt{3}}{\cancel{2}}\right)$$

Cancel where applicable.

$$= -\pi + 2\sqrt{3}$$

Then combine like terms.

The challenging part of the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3}4\cos^2\theta\,d\theta = 2\int_{-\pi/3}^{\pi/3}\cos^2\theta\,d\theta$$

This part of the integral is more challenging because it has cosine squared, but no sine term to allow a *u*-substitution.

$$= \cancel{2}\int_{-\pi/3}^{\pi/3}\frac{\cos 2\theta+1}{\cancel{2}}\,d\theta = \int_{-\pi/3}^{\pi/3}(\cos 2\theta+1)\,d\theta$$

Use the cool substitution that comes from the double-angle formula.

$$= \frac{\sin(2\theta)}{2}+\theta\,\Big|_{-\pi/3}^{\pi/3} = \frac{\sin(2\pi/3)}{2}+\frac{\pi}{3}-\left[\frac{\sin(-2\pi/3)}{2}-\frac{\pi}{3}\right]$$

Integrate with respect to θ and evaluate the result at the limits of integration.

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)+\frac{\pi}{3}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)+\frac{\pi}{3} = \frac{\sqrt{3}}{2}+\frac{2\pi}{3}$$

Combine like terms.

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3}[-3+4\cos\theta\,]\,d\theta + \frac{1}{2}\int_{-\pi/3}^{\pi/3}4\cos^2\theta\,d\theta$$

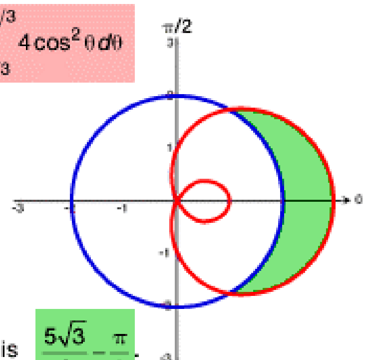
$$= -\pi+2\sqrt{3} + \frac{\sqrt{3}}{2}+\frac{2\pi}{3}$$

Add the results of the two integrals you evaluated.

$$= -\frac{\pi}{3}+\frac{5\sqrt{3}}{2}$$

$$= \frac{5\sqrt{3}}{2}-\frac{\pi}{3}$$

Combine like terms.



The area of the shaded crescent is $\dfrac{5\sqrt{3}}{2}-\dfrac{\pi}{3}$.

By setting up an integral, choosing appropriate endpoints and using the necessary techniques to evaluate the integral, you can determine the area of a region bounded by two polar curves.