# Calculus Lecture Notes

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**Unit:** Elementary Functions and Their Inverses **Module:** Inverse Functions

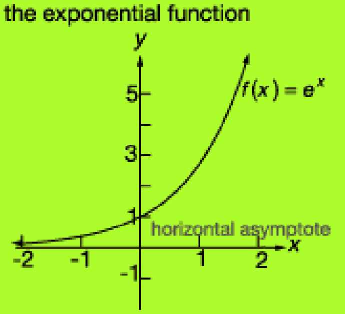
## The Exponential and Natural Log Functions

**key concepts:**

- Raising the number $e$ to the power $x$ produces the **exponential function**. It is a positive, increasing function.
- Taking the log to the base $e$ of $x$ $(\log_e x)$ produces the **natural log function,** noted $\ln x$. It is an increasing function defined only for positive $x$-values.
- The exponential and natural log functions are **inverses** of each other.

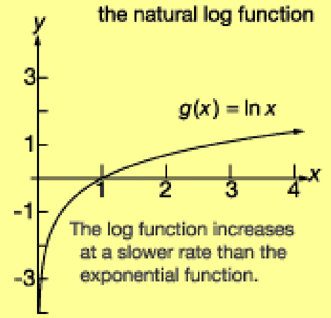
### Reliving the exponential and natural log functions

the exponential function




The exponential function increases very quickly as $x$ increases.

the natural log function




The natural log function is defined only for positive values.

On the left, the graph of the **exponential function** ($e^x$) shows that it is positive and increasing. The $x$-axis is a left horizontal asymptote for the curve at $-\infty$.

On the right, the graph of the **natural log function** ($\ln x$) shows that it is increasing. It is only defined for positive $x$-values, in contrast to the exponential function. The $y$-axis is a vertical asymptote for the curve at $-\infty$.

$e^{\ln x} = x$

$\ln(e^x) = x$

Notice that these functions undo each other, leaving you with just $x$.

The exponential function and the natural log function are inverse functions of each other.

$$\begin{aligned} \ln x &= \int_1^x \frac{1}{t}\,dt \\ &= \ln|t|\,\Big|_1^x \\ &= \ln x - \ln 1 \\ &= \ln x - 0 \\ &= \ln x \end{aligned}$$

The exponential and natural log functions are **inverses** of each other. When you compose them in either order, they cancel each other out.

Some mathematicians define $\ln x$ as the definite integral from one to $x$ of the function $1/t$.

### Reliving the derivatives

$$\frac{d}{dx}\left[e^x\right] = e^x$$

The derivative of the exponential function is itself.

**IT IS e-GOTISTICAL.**

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

The derivative of the natural log function is $\frac{1}{x}$.

The derivative of $e^x$ is $e^x$.

The derivative of $\ln x$ is $1/x$.

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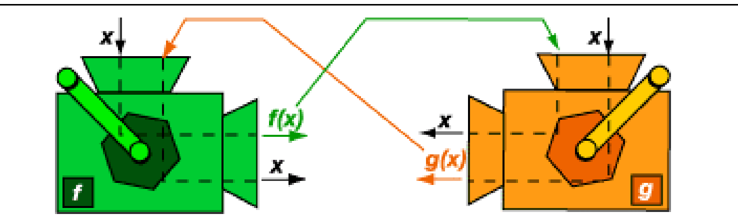
# The Basics of Inverse Functions

key concepts:

- **Inverse functions** undo each other.
- In inverse functions, the **dependent variable** and **independent variable** switch roles. The graph of an inverse function looks like a mirror reflection of the original graph.
- Functions that are not one-to-one do not have inverses. **One-to-one functions** pass both the **vertical line test** and the **horizontal line test**.




Inverse function notation

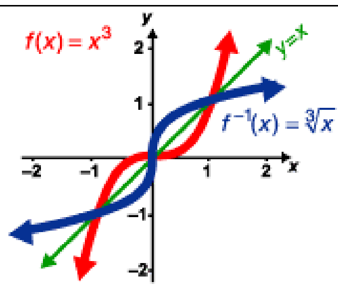
$g(x) = f^{-1}(x)$ opposite of $f$

$f^{-1}(x)$ does not mean $\frac{1}{f(x)}$.

A function $f$ is like a machine which takes a number $x$ and cranks out another number, $f(x)$.

It can be helpful to have a machine that reverses the process of the first machine. That machine is called the **inverse function** of the original function.

The inverse of a function $f$ is noted by a raised –1. Do not confuse this with an exponent of –1, which symbolizes the reciprocal.




An inverse function switches the roles of $x$ and $y$.

To do this graphically we can flip the graph across the line $y=x$.

The blue curve is a mirror reflection of the red curve.

The blue curve is the inverse of the red curve.

Graphically the inverse function is a reflection of the original curve across the line $y=x$.

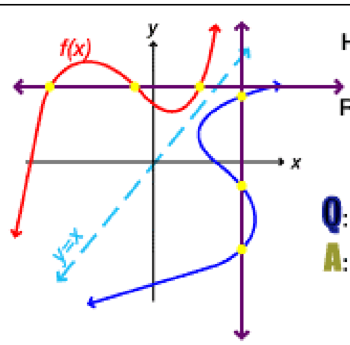
Algebraically, $f^{-1}(x) = \sqrt[3]{x}$

$$f^{-1}(f(x)) = f^{-1}(x^3) \quad \text{Find } f(x).$$
$$= \sqrt[3]{(x^3)} \quad \text{Evaluate } f^{-1} \text{ at } x^3.$$
$$= x$$

If you have a function that relates two variables, $x$ and $y$, then the inverse function will switch them.

You can make the switch graphically by reflecting the first graph across the line given by $y = x$.

You can verify algebraically that $f$ and $f^{-1}$ are inverses of each other by composing them. Both $f^{-1}(f(x))$ and $f(f^{-1}(x))$ should equal $x$.




How do you know if a function has an inverse?

Reflect the curve to find the inverse, but this reflection might not be a function.

**Q**: Is the reflected image a function?

**A**: No, it fails the vertical line test.

So, the function is not invertible.

If the reflected image of a function does not pass the **vertical line test**, then it is not a function. Therefore the inverse does not exist.

You can see that if the curve of the original function (on the left) does not pass the **horizontal line test**, then its reflection (on the right) will not pass the vertical line test.

If a function is strictly increasing or strictly decreasing, then it is **one-to-one**.

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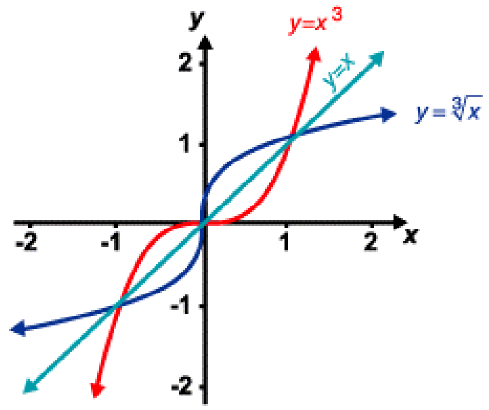
## Finding the Inverse of a Function

key concepts:

- To determine the **inverse** of a function algebraically, swap the **independent variable** ($x$) and the **dependent variable** ($y$) and then solve for $y$.
- Verify the inverse by composing it with the original function as described in the definition of an inverse.




To find the **inverse** of a function graphically, you reflect the curve of the function across the line given by $y = x$.

This reflection just swaps the roles of $y$ and $x$.

**Example!** Find the inverse of :

$$f(x) = 5x^3 + 1$$

$$y = 5x^3 + 1$$

$$x = 5y^3 + 1$$ Swap the roles of $x$ and $y$.

$$x - 1 = 5y^3$$ Subract one from both sides.

$$\frac{\cancel{5}y^3}{\cancel{5}} = \frac{x-1}{5}$$ Divide both sides by five.

$$y^3 = \frac{x-1}{5}$$

$$y = \sqrt[3]{\frac{x-1}{5}}$$ Take the cube root of both sides.

So the inverse is $\boxed{f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}.}$

To find the inverse of a function algebraically, first rename the function $y$. Then swap $y$ and the independent variable, which is usually $x$.

Solve for $y$ to get an expression for the inverse function.

Since the original function is $f(x)$, the inverse is noted as $f^{-1}(x)$.

If $f(x) = 5x^3 + 1$ then $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$.

So we know that $f(f^{-1}(x)) = x$.

But to check that two functions are inverses of each other you must check both directions.

$$f^{-1}(f(x)) = f^{-1}(5x^3 + 1)$$ Find $f(x)$.

In order for two functions to be inverses of each other, all roads must lead to $x$.

Evaluate the compositions $f(f^{-1}(x))$ and $f^{-1}(f(x))$ to make sure they both equal $x$. You must check both directions.