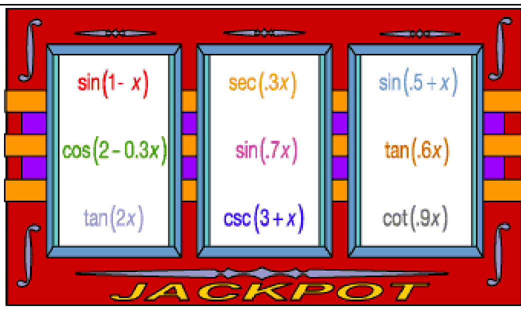
Unit: Techniques of Integration **Module:** Integrals Involving Powers of Other Trig Functions

Integrals of Other Trigonometric Functions

key concepts:

- Integrate tangent and cotangent by expressing them in terms of sine and cosine and then using ***u*-substitution**.
- Integrate secant and cosecant by multiplying by an expression equal to one.



Although many trigonometric functions can be integrated, it is not true that all trig integrals can be evaluated.

If you randomly combine different trig factors together, it isn't likely that the result will be integrable. Those that are usually require some modification before techniques of integration will work on them.

Q What is the integral of tan x?

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$ Convert back to cos x.

$$= \ln\left(|\cos x|^{-1}\right) + C$$ This expression is fine, but you can simplify it.

$$\boxed{= \ln|\sec x| + C}$$

A $\int \tan x\,dx \boxed{= \ln|\sec x| + C}$

Similarly $\int \cot x\,dx = \ln|\sin x| + C$

The integrals of sine and cosine are usually covered in Calculus I. But did you ever wonder what the integral of tangent is? To solve this integral, start by expressing tangent in terms of sine and cosine.

Now you can use ***u*-substitution** to solve the integral.

A lot of mathematicians do not like negative signs in the answers to their integrals. Notice that by using a log property you can move the negative into the exponent. Then you can replace the cosine term with a secant.

A similar method can be used to find the integral of cotangent.

Evaluate $\int \sec x\,dx$.

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$$ Multiply by a fraction equal to one.

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$\boxed{= \ln|\sec x + \tan x| + C}$$

Let $u = \sec x + \tan x$

$du = \left(\sec x \tan x + \sec^2 x\right)dx$

Expressing secant in terms of cosine doesn't leave a term for the du in a u-substitution. Instead, try multiplying secant by some expression equal to one.

This integral might look more complicated, but it can be evaluated by u-substitution. The numerator is exactly the derivative of the denominator.

Don't forget to express your final answer in terms of x and not u.

Notice that evaluating trig integrals involves knowing a few tricks. The more of these tricks you see, the more integrals you can solve.

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Integrals with Odd Powers of Tangent and Any Power of Secant

key concepts:

- Steps for integrating odd powers of tangent:
  1. Factor out a tangent.
  2. Use the form of the **Pythagorean identity** involving tangent.
  3. Use *u*-substitution to evaluate the integral.

- Steps for integrating an odd power of tangent with any power of secant:
  1. Factor out a tangent and a secant.
  2. Use the form of the Pythagorean identity involving tangent.
  3. Use *u*-substitution to evaluate the integral.

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x = \sec^2 x - 1$

forms of the Pythagorean identity

Evaluate $\int \tan^3 x\,dx$.

$$\int \tan^3 x\,dx = \int \tan^2 x \tan x\,dx$$ Factor out tangent.

$$= \int \left(\sec^2 x - 1\right)\tan x\,dx$$ Use the new form of the Pythagorean identity.

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$= \int u\,du - \int \tan x\,dx$$

$$= \frac{u^2}{2} - \int \tan x\,dx$$

$$\boxed{= \frac{\tan^2 x}{2} - \ln\left|\sec x\right| + C}$$

Let $u = \tan x$

$du = \sec^2 x\,dx$

Divide the **Pythagorean identity** by $\cos^2 x$ to get a new identity in terms of secant and tangent.

This identity is very powerful for solving integrals with odd powers of tangent.

Start by factoring out a single tangent.

Now use the Pythagorean identity to transform two of the remaining tangents into secants.

The integral can now be separated into easier integrals.

Notice that the first integral is solvable with a *u*-substitution.

The integral of tangent is also solvable with a *u*-substitution, setting *u* equal to cos *x*.

Don't forget the constant of integration.

Consider $\int \tan^3 x \sec^5 x\,dx$.

$$\int \tan^3 x \sec^5 x\,dx = \int \tan^2 x \sec^4 x \sec x \tan x\,dx$$ Factor out tangent and secant.

$$= \int \left(\sec^2 x - 1\right)\sec^4 x\left(\sec x \tan x\right)dx$$

$$= \int \left(u^2 - 1\right)u^4\,du$$

$$= \int u^6 - u^4\,du$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$\boxed{= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C}$$

Let $u = \sec x$

$du = \sec x \tan x\,dx$

In cases where the integral combines odd powers of tangent with secant, the same identity can be used.

Start by factoring out a secant and a tangent.

Now transform any remaining tangents into secants using the Pythagorean identity.

Now make a *u*-substitution setting *u* equal to sec *x*.

Notice that the integral is a little different from integrals involving strictly tangent because the *u*-substitution is occurring within a single integral.

Now you can integrate using the power rule.

Integrals involving tangents and secants require some additional steps, but by using the Pythagorean identity you can conquer them.

Calculus Lecture Notes

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Integrals with Even Powers of Secant and Any Power of Tangent

key concepts:

- Steps for integrating an even power of secant with any power of tangent:
  1. Factor out a secant-squared.
  2. Use the form of the **Pythagorean identity** involving secant.
  3. Use *u*-substitution to evaluate the integral.

The following technique will work anytime an integral is made up strictly of tangents and secants and the secant is raised to an even power.

An even power of secant with any power of tangent

$$\boxed{\int \tan^4 x \sec^6 x\,dx} = \int \tan^4 x \sec^4 x \sec^2 x\,dx \quad \text{Factor out } \sec^2 x.$$

$$= \int \tan^4 x \left(\tan^2 x + 1\right)^2 \sec^2 x\,dx$$

$$= \int u^4 \left(u^2 + 1\right)^2 du$$

$$= \int u^4 \left(u^4 + 2u^2 + 1\right) du$$

$$= \int \left(u^8 + 2u^6 + u^4\right) du \quad \text{Distribute.}$$

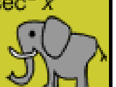
$$= \frac{u^9}{9} + \frac{2u^7}{7} + \frac{u^5}{5} + C$$

$$\boxed{= \frac{\tan^9 x}{9} + \frac{2\tan^7 x}{7} + \frac{\tan^5 x}{5} + C}$$

$\tan^2 x = \sec^2 x - 1$

$\tan^2 x + 1 = \sec^2 x$

Remember



Let $u = \tan x$

$du = \sec^2 x\,dx$

To solve such an integral, start by setting aside two of the secant factors.

Convert all of the remaining terms into cosines using a modified version of the **Pythagorean identity**.

Now the integral can be solved with a *u*-substitution.

Remember to express your answer in terms of *x* and to add the constant of integration.

The power of *u*-substitution and the Pythagorean identity now enable you to solve hundreds of integrals you couldn't solve before.