Unit: Elementary Functions and Their Inverses **Module:** Calculus of Inverse Trig Functions

Derivatives of Inverse Trigonometric Functions

key concepts:

- To find the derivative of an inverse trig function, rewrite the expression in terms of standard trig functions, differentiate implicitly, and use the Pythagorean theorem

Finding the derivatives of inverse trig functions

Given $x = \sin y$, find $\frac{dy}{dx}$. Use implicit differentiation to find the derivative.

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin \bigcirc)$$

Use the chain rule.

$$1 = (\cos \bigcirc) \cdot \frac{d}{dx}(\bigcirc)$$

$$1 = (\cos y) \cdot \frac{d}{dx}(y)$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

Given $x = \sin y$, find $\frac{dy}{dx}$. Express the derivative in terms of x. $\frac{dy}{dx} = \frac{1}{\cos y}$

The Pythagorean theorem: $a^2 + b^2 = c^2$

$$(??)^2 + x^2 = 1$$

Use the Pythagorean theorem.

$$(??)^2 = 1 - x^2$$

$$\sqrt{(??)^2} = \sqrt{1-x^2}$$

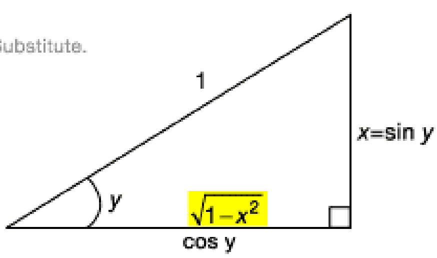
Take the square root.

$$(??) = \sqrt{1-x^2}$$

$$\cos y = \sqrt{1-x^2}$$

Substitute.

$$\frac{dy}{dx} = \frac{1}{\cos y} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$



To find the derivative of arcsin x, first think of it as $y = \arcsin x$. Then rewrite the expression using standard trigonometric functions.

Use implicit differentiation to take the derivative of both sides.

Remember to use the chain rule. The derivative of y with respect to x is dy/dx.

The result you get by differentiating is in terms of y, but you want it in terms of x.

Let y be an angle of a right triangle. Since $\sin y = x$, you can let the opposite side equal x and the hypotenuse equal one.

Using the Pythagorean theorem, you can write cos y in terms of x. Now you have an expression for the derivative of arcsin x.

It may seem strange that the derivative is not in terms of any of the other trig or inverse trig functions. If you remember that the trigonometric functions are all defined by right triangles, then the derivative makes more sense.

The derivatives of inverse trig functions

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

If you use the same method as above, you can determine the derivatives of all the inverse trig functions.

These derivatives may be difficult to memorize. But if you remember the method, then you can always derive them again.

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More Calculus of Inverse Trigonometric Functions

key concepts:

- Differentiation techniques such as the chain rule and product rule can be applied to derivatives involving inverse trigonometric functions.
- Working backwards from the derivative formulas for inverse trig functions produces integration formulas.

Inverse trig and the chain rule

Todd$(x) = \arctan(3x)$, find Todd$'(x)$.

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

$$\text{Todd}'(x) = \frac{1}{1+(\quad)^2}\cdot(\quad)'$$

Use the chain rule and the derivative formula for the arctangent.

$$= \frac{1}{1+(3x)^2}\cdot 3$$

Plug in the and the derivative of the

$$\boxed{= \frac{3}{1+9x^2}}$$

Expand the squared-term and simplify.

Although you often hear of functions named *f*, they can also be named *g* or anything else you want, even "Todd". The name doesn't matter; the function is still just a function.

Because Todd(x) involves arctangent of a function of x, you need to use the chain rule on the inside part.

Just substitute into the formula for the derivative and then multiply by the derivative of the inside.

Inverse trig and integration

Using the derivative formulas backwards gives you the anti-derivatives for some expressions you couldn't integrate before.

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\int \frac{1}{1+x^2}\,dx = \arctan x + C$	$\int \frac{-1}{1+x^2}\,dx = \operatorname{arccot} x + C$
$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C$	$\int \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \operatorname{arccsc} x + C$

These formulas may look familiar. They're the same formulas as for the derivatives of inverse trig functions, but expressed as antiderivatives.

$$\int \frac{-1}{\sqrt{5-5x^2}}\,dx = \int \frac{-1}{\sqrt{5(1-x^2)}}\,dx$$

The arcosine function is the closest match.

$$= \int \frac{1}{\sqrt{5}}\cdot\frac{-1}{\sqrt{(1-x^2)}}\,dx$$

Factor out the 5.

$$= \frac{1}{\sqrt{5}}\int \frac{-1}{\sqrt{(1-x^2)}}\,dx$$

Since $1/\sqrt{5}$ is a constant, move it outside of the integral.

$$\boxed{= \frac{1}{\sqrt{5}}(\arccos x) + C}$$

Now use the formula to integrate the cosine.

Don't forget the constant of integration.

None of the integral formulas match with this integral. However, the arccosine formula comes close. Factor the five out of the radicand, and then use the formula.

Mathematics is about identifying patterns. If the underlying pattern matches the integration formula, try to transform the integrand by factoring until you can apply the formula.