Calculus Lecture Notes

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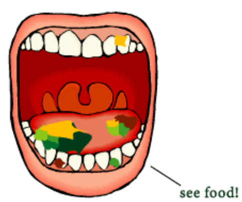
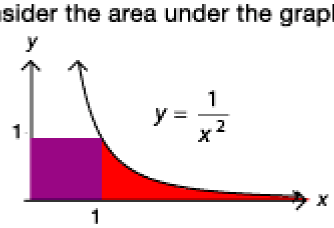
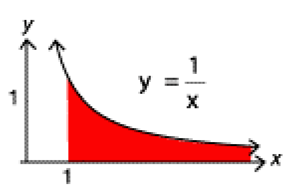
Unit: Improper Integrals **Module:** Convergence and Divergence of Improper Integrals

The First Type of Improper Integral

key concepts:

- An **improper integral** is a definite integral with one of the following properties: the integration takes place over an infinite interval or the integrand is undefined at a point within the interval of integration.
- With some improper integrals, the area of the region under the curve is finite even though the region extends to infinity.
- An improper integral that is infinite diverges. An improper integral that equals a numerical value converges.

Q: Can integrals be improper?  A: Yes.	A definite integral is considered an **improper integral** if it has one of these properties: • the integration is over an infinite interval, or • its integrand is undefined at a point within the interval of integration.
Consider the area under the graph of $y = \frac{1}{x^2}$.  $"\int_1^{\infty} \frac{1}{x^2}\,dx" = "\frac{-1}{x}\Big\|_1^{\infty}"$ $= 0 - (-1)$ $\boxed{=1}$	Imagine calculating the area of the region under the curve $y = \frac{1}{x^2}$ starting at $x = 1$ and moving to the right. You would integrate the function from 1 to ∞ and solve the improper integral. The solution of the integral is one. This means that the area under the curve to the right of $x = 1$ has an area of one. You can think about this area "repackaged" into the square bounded by the origin and the point (1,1). Because this improper integral has a finite value, it converges.
Starting at $x = 1$, determine the area under the graph of $y = \frac{1}{x}$.  $\int_1^{\infty} \frac{1}{x}\,dx = \ln x\Big\|_1^{\infty}$ $= \infty - 0$ $= \infty$	An improper integral diverges if its value is infinite. Imagine calculating the area under the curve $y = \frac{1}{x}$ starting at $x = 1$ and moving to the right. You would integrate the function from 1 to ∞ and solve the improper integral. In this example, the value of the integral is infinity, meaning that the area under the curve is infinitely large. This improper integral diverges.

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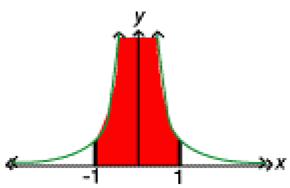
The Second Type of Improper Integral

key concepts:

- If a function is not continuous on the integration interval, then the standard procedure will not work. Use the discontinuity as an endpoint for the integral. This is the second type of **improper integral**.

Q: Can integrals be improper?  A: Yes.	Remember: A definite integral is considered an **improper integral** if it has one of these properties: • the integration is over an infinite interval, or • its integrand is undefined at a point within the interval of integration.
Q: What is the area under the curve of $y = \frac{1}{x^2}$ between -1 and 1? 	A second type of **improper integral** occurs when the function is not continuous over the interval of integration. In this example, the function is undefined and the curve is discontinuous at $x = 0$. This makes the integral improper. In order to get the correct answer, you have to set up the integral so that the point of discontinuity is an endpoint.
$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right\rvert_{-1}^{1}$ $= -1 - (1)$ $= -2$	If you didn't realize that the function is discontinuous at $x = 0$, you might set up the problem with limits of integration between negative one and one. This would result in a solution with the area under the curve equal to negative two. This answer is incorrect. Notice that the function is squared so it is positive everywhere; the correct answer must also be positive.
REAL AREA $2\int_{0}^{1} \frac{1}{x^2}\,dx = 2\left(\frac{-1}{x}\right)$ $= \left.\frac{-2}{x}\right\rvert_{0}^{1}$ $= -2 - (-\infty)$ $= \infty$	To set up the integral correctly, use the point of discontinuity, $x = 0$, as a limit of integration. Because the function $\frac{1}{x^2}$ is symmetric about the y-axis, find the area under the curve between zero and one and then double the result. Evaluating the antiderivative, $\frac{-2}{x}$, as $x \to 0$ results in $-\infty$. This **improper integral** diverges.

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Infinite Limits of Integration, Convergence, and Divergence

key concepts:

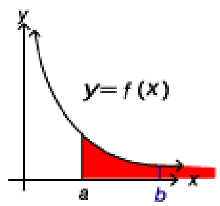
- **Improper integrals** can be expressed as the limit of a proper integral as some parameter approaches either infinity or a discontinuity.

Formalizing the idea of **improper integrals** involves replacing the infinite endpoint with a parameter whose limit approaches either infinity or the discontinuity.

A formal idea of improper integrals over an infinite interval

$$A = \int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to \infty} \int_a^b f(x)dx$$



By choosing a midpoint, *t*, you can split this last integral.

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{c \to \infty} \int_c^t f(x)dx + \lim_{d \to \infty} \int_t^d f(x)dx$$

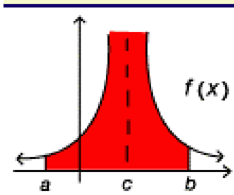
There are three types of **improper integrals** over an infinite interval:

1. The right endpoint is infinite. To formalize this integral ∞ is replaced with b and the integral is evaluated as $b \to \infty$.

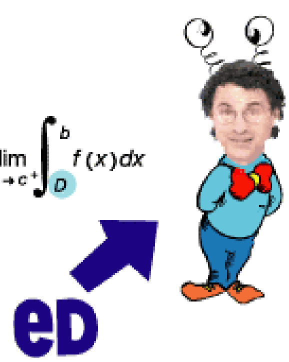
2. The left endpoint approaches negative infinity. To formalize this integral ∞ is replaced with a and the integral is evaluated as $a \to -\infty$.

3. The range of integration is the entire *x*-axis. Split the integral into the sum of two integrals each of which has a limit if integration at some midpoint, *t*. The fist integral can be evaluated as in example 2 above and the second can be evaluated as in example 1.

A formal idea of improper integrals over a discontinuity



$$\int_a^b f(x)dx = \lim_{E \to c^-} \int_a^E f(x)\,dx + \lim_{D \to c^+} \int_D^b f(x)dx$$



In this case the integral is improper because its domain has a discontinuity. Split the integral into the sum of two integrals each of which has a limit of integration at the discontinuity, $x = c$. The fist integral is formalized by replacing c with E and evaluating the integral as $E \to c^-$. (Note that c^- means E approaches c from the left or negative side of the *x*-axis.) The second integral is formalized by replacing c with D and evaluating the integral as $D \to c^+$. (c^+ indicates that D approaches c from the right or positive side of the *x*-axis.)