Unit: Techniques of Integration **Module:** Integration by Parts

Introduction to Integration by Parts

key concepts:

- **Integration by parts** reverses the **product rule**.
- Integration by parts requires a product of two functions: one with a simple derivative and one with a simple integral.

Consider $\int \ln x\, dx$.

Q Could you use a previous technique to evaluate this integral?

✗ *U*-substitution will not work.

✗ The integrand does not involve a fraction, so partial fraction decomposition does not apply.

A No, nothing seems to apply.

The integral of ln$x$ is a tough nut to crack. The techniques of integration that you've studied, u-substitution and partial fraction decomposition, don't apply. A new technique must be used. This technique is called **integration by parts**.

PRODUCT RULE $\quad (f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$

$$(f(x)g(x))' - f(x)g'(x) = g(x)f'(x)$$

$$\int\left[(f(x)g(x))' - f(x)g'(x)\right]dx = \int g(x)f'(x)\,dx$$

$$\boxed{f(x)g(x) - \int f(x)g'(x)\,dx = \int g(x)f'(x)\,dx}$$

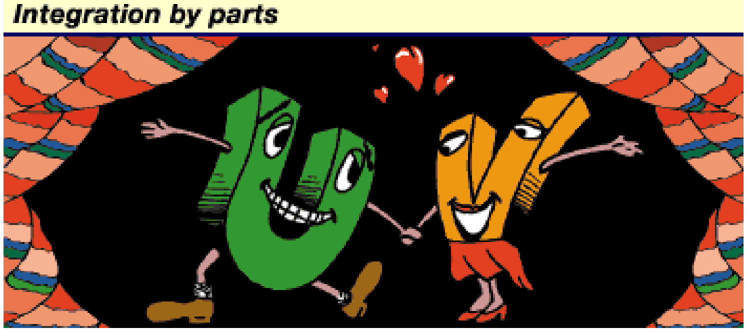
differentiable integrable

Recall the **product rule**. It is used to differentiate the product of two functions.

Subtract $f(x)g'(x)$ from both sides.

Integrate both sides. Note that one antiderivative of $[f(x)g(x)]'$ is $f(x)g(x)$. The equation in the box is the formula for integration by parts.

Integration by parts



You can rewrite the integration by parts formula as follows

$$\int u\,dv = uv - \int v\,du$$

differentiable integrable

Use integration by parts to integrate a function that has two parts, one part that is easy to differentiate and one part that is easy to integrate.

The formula for integration by parts is often rewritten so that $g(x) = u$ and $f(x) = v$. In this case, u is easily differentiable and dv is easily integrable. When written this way, integration by parts can be thought of as a love affair between the two functions, u and v.

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Applying Integration by Parts to the Natural Log Function

key concepts:

- **Integration by parts** reverses the product rule.
- Integration by parts requires a product of two functions: one with a simple derivative and one with a simple integral.
- Integration by parts: $\int u\,dv = uv - \int v\,du$.

The cast of characters

INSPIRATIONAL EXAMPLE

Evaluate $\int \ln x\,dx$.

Identify two factors, one which is simple to integrate and one which is simple to differentiate.

$$\int \ln x\,dx = \int 1\cdot \ln x\,dx$$

Reminder: One times any expression yields that expression.

$u = \ln x$ $\quad v = x$

$du = \frac{1}{x}dx$ $\quad dv = 1dx$

The Cast of Characters

For u, choose the factor which is simple to differentiate.

For dv, choose the factor which is simple to integrate.

$$\int 1\cdot \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$$

Substitute the terms into the formula.

$$= x\ln x - \int 1dx$$

Simplify the integrand.

$$= \boxed{x\ln x - x + C}$$

INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du$$

A function you often encounter in Calculus is the natural log function. But how do you integrate it?

Notice that the integral is made up of two pieces, the piece with the log and the piece without.

To use **integration by parts** you have to decide which parts match up in the formula.

For u, pick the part that is easy to differentiate, like the natural log. For dv, pick the part that is easy to integrate.

Once you have matched the parts up then differentiate or integrate to find the other parts you need. Plug all of these parts into the formula.

Look for ways to simplify the new integrand.

Here is the integration by parts formula color-coded to match the pieces in the evaluated integral.

Checking the answer

INSPIRATIONAL EXAMPLE

Evaluate $\int \ln x\,dx$.

$$\boxed{x\ln x - x + C}$$

PRODUCT RULE

$$\frac{d}{dx}(x\ln x - x + C) = x\left(\frac{1}{x}\right) + \ln x - 1$$

Use the product rule.

$$= 1 + \ln x - 1$$

Simplify the expression.

$$= \ln x$$

Reminder: Check the result of integration by differentiating.

What if you aren't convinced that integration by parts gave you the right answer? Remember that you can always check the results of integration by taking the derivative.

To evaluate this derivative you have to use the product rule. That is not surprising since integration by parts undoes the product rule.

You get back what you started with. So the technique gave you the right answer!

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Inspirational Examples of Integration by Parts

key concepts:

- **Integration by parts** requires a product of two functions, one with a simple derivative and one with a simple integral.
- Integration by parts: $\int u\,dv = uv - \int v\,du$.

An integrand with an exponential factor

Inspirational Example

Evaluate $\int xe^x\,dx$.

- ✗ U-substitution will not work.
- ✓ Since the integrand is a product, the integral is a candidate for integration by parts.
- ✓ Both factors are simple to integrate or differentiate.

The Cast of Characters

$$u = x \qquad v = e^x$$

$$du = dx \qquad dv = e^x\,dx$$

Since the derivative and the integral of e^x are both e^x, you can substitute it for u or dv.

The derivative of x is one, which is less complicated than its integral.

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$\boxed{= xe^x - e^x + C}$$

Tip: If your replacements for u and dv make the integral more difficult, just try switching them.

Integration by parts can be applied to functions with exponential factors. Consider this example.

Notice that the integrand is easily broken into two factors. The exponential factor is easy to differentiate or integrate. The linear factor is easier to differentiate than integrate. So let the linear factor equal u.

Since you know u, just differentiate to find du.

Since you know dv, just integrate to find v.

Once you have each of the parts you can substitute them into the integration by parts formula.

Notice that the integral of the exponential function is very easy to evaluate. If your new integral is harder to integrate, you might want to go back and switch your choices for u and dv.

An integrand with a logarithmic factor

Inspirational Example

Evaluate $\int x\ln x\,dx$.

- ✓ Integrating $\ln x$ is complicated, while differentiating it is quite simple.
- ✓ The integral of x is straightforward.

The Cast of Characters

$$u = \ln x \qquad v = \frac{x^2}{2}$$

$$du = \frac{1}{x}\,dx \qquad dv = x\,dx$$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$$

$$\boxed{= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C}$$

Tip — PRODUCT RULE: Check your work by finding the derivative using the product rule.

You have already seen that integration by parts can evaluate the integral of the natural log function. But what if you are asked to integrate the natural log function times a linear factor?

Integration by parts will still solve this integral.

Since the logarithmic function is easy to differentiate, let it equal u. Set the other factor equal to dv and plug into the formula.

Notice that you never have to integrate the natural log function. The v-term and the du-term cancel leaving a simple linear term to integrate.

You can always check your work when you use integration by parts. Just find the derivative through the power rule.

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Repeated Application of Integration by Parts

key concepts:

- **Integration by parts**: $\int u\,dv = uv - \int v\,du$.
- Using integration by parts once might not be enough to evaluate an integral. As long as integration by parts simplifies the factors of the integral, it is not a bad idea to use the technique again. You can repeat applications as long as there are factors.

An integrand with a trigonometric factor

Evaluate $\int x^2 \sin x\,dx$. INSPIRATIONAL EXAMPLE

Here is an integral with a quadratic and a trigonometric factor. Try to use **integration by parts** on this integral.

✓Both factors are simple to integrate or differentiate.

$u = x^2 \qquad v = -\cos x$

$du = 2x\,dx \qquad dv = \sin x\,dx$

The Cast of Characters

!Notice that the derivative and the integral of $\sin x$ have the same complexity.

!Differentiating x^2 reduces the complexity.

Both factors are easy to integrate or differentiate, but integrating the quadratic makes it more complicated than differentiating. So let the quadratic factor equal u.

$$\int x^2 \sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$$

$$= -x^2\cos x + \int 2x\cos x\,dx$$

Apply the formula for integration by parts.

Plug the pieces into the integration by parts formula.

The new integral looks a little easier than the original, but it still cannot be evaluated with u-substitution or any other obvious method. What technique of integration would work well here?

Notice that the original integrand has a second degree factor, and the new one has a first degree factor. You have reduced the complexity.

Notice that the new integral is made up of a trigonometric factor and a linear factor. That makes it a good candidate for integration by parts. But can you use integration by parts again if you already used it once? Sure you can! Think of the new integral as its own sub-problem.

ASIDE $\int x\cos x\,dx$

$u = x \qquad v = \sin x$

$du = 1\cdot dx \qquad dv = \cos x\,dx$

The Cast of Characters

Since the linear term becomes less complex when you differentiate it, set it equal to u.

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

Integration by parts gives you the answer to the new integral.

$$\int x^2 \sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$$

$$= -x^2\cos x + \int 2x\cos x\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2(x\sin x + \cos x) + C$$

$$= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$$

Don't forget the constant.

Now you can go back to the original problem and substitute in the information you found.

Notice that the entire integral is multiplied by that factor of two outside the parentheses.

As long as the integration by parts produces factors of lower complexity, you should feel comfortable applying integration by parts again.

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Algebraic Manipulation and Integration by Parts

key concepts:

- **Integration by parts**: $\int u\,dv = uv - \int v\,du$.
- Sometimes integration by parts will yield an expression with the original integral on both sides of the equal sign. Solving algebraically yields the integral.

Integration by parts magic

EXAMPLE

Evaluate $\int e^{2x}\sin(3x)\,dx$. Original

Both factors are simple to integrate or differentiate, but neither one changes the complexity of the integrand.

$u = e^{2x}$ $\qquad v = -\frac{1}{3}\cos(3x)$

$du = 2e^{2x}\,dx$ $\qquad dv = \sin(3x)\,dx$

The Cast of Characters

$$\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) - \int -\frac{2}{3}e^{2x}\cos(3x)\,dx$$

$$= \boxed{-\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)\,dx} \quad (1)$$

Apply the formula for integration by parts.

Factor out the constant.

$u = e^{2x}$ $\qquad v = \frac{1}{3}\sin(3x)$

$du = 2e^{2x}\,dx$ $\qquad dv = \cos(3x)\,dx$

$$\int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\,dx \quad (2)$$

$$I = \int e^{2x}\sin(3x)\,dx$$

$$I = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}I\right)$$

$$\frac{9}{9}I = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}I$$

Distribute.

$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)$$

$$\int e^{2x}\sin(3x)\,dx = \frac{9}{13}\left[-\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)\right] + C$$

Solve for I.

Original

Consider this integral. Both of the factors are easy to integrate and differentiate. But notice that neither process makes the factors any simpler.

So you don't have an idea about which factor to use where. But it is better to keep working than to stare at the problem and make no progress. So go ahead and pick one for u and move forward.

Notice that the resulting integral is still rather complicated. But you can reuse integration by parts.

Watch carefully here! There's a magic trick.

Try to evaluate the new integral.

Pick the same piece for u.

Use the integration by parts formula and plug in the pieces.

Notice that the resulting integral for the second piece includes an integral that exactly matches the original question.

So set the original integral equal to I.

Notice that if you plug the result from the sub-problem back into the original problem, you can isolate the I-terms.

So in a way you can evaluate this integral without ever taking an integral. That's magic!