Unit: Applications of Integration **Module:** Finding Volumes Using Cross-Sections

Finding Volumes Using Cross-Sectional Slices

key concepts:

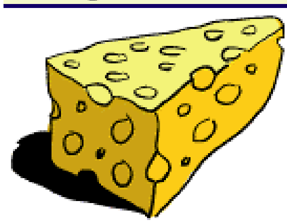
- The volume of a solid with **vertical cross-sections** of area $A(x)$ is V where:

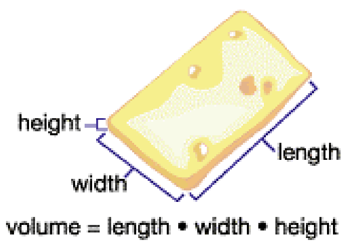
$$V = \int_a^b A(x)dx \, .$$

- The volume of a solid with **horizontal cross-sections** of area $A(y)$ is V where:

$$V = \int_a^b A(y)dy \, .$$

Slicing cheese to find its volume





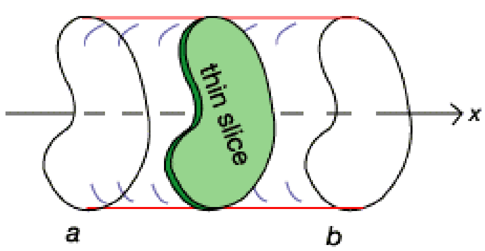
Finding the volume of an object is pretty easy if you know the formula. But a lot of objects don't fit any commonly known formulas. How do you find the volumes of these strangely shaped objects?

If you consider the object in slices, then all you would have to do is find the volume of each individual slice and then add them together. These slices are called **cross-sections**.

This process is actually a pretty good way to calculate the volume of an object when the cross-sections are easily defined. Consider a hunk of cheese. If you slice it up, the individual slices look like rectangular prisms.

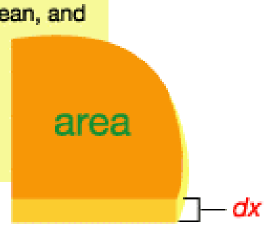
Finding the volume of a solid using vertical slices

What is the volume (V) of this region?



Let V be the volume of the cylindrical lima bean, and let $A(x)$ be the area of the face of each slice.

$$V = \int_a^b A(x)dx$$



tip: Integrating area yields volume.

Finding volumes is one of the applications of the definite integral.

Suppose you are given an object and you can define the cross sections by a function in terms of x.

The volume of the object on the interval $[a,b]$ is given by the definite integral to the left. All the integral does is sum up an infinite number of slices of the volume, each of a very tiny width called dx.

The same process can be applied to an object whose cross sections are defined by a function in terms of y.

An Example of Finding Cross-Sectional Volumes

key concepts:

- The volume of a solid with **vertical cross-sections** of area $A(x)$ is V where:

$$V = \int_a^b A(x)dx\ .$$

- The volume of a solid with **horizontal cross-sections** of area $A(y)$ is V where:

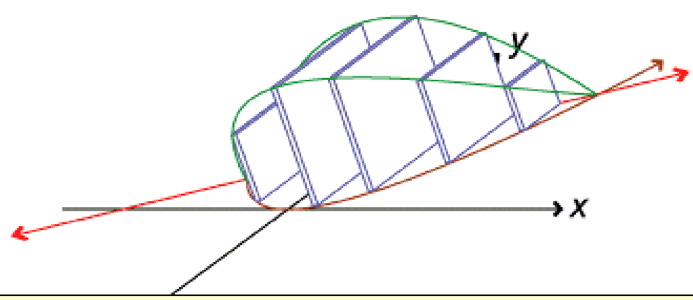
$$V = \int_a^b A(y)dy\ .$$

Determining the shape of a solid

Consider a solid whose base is the region bounded by the graphs of: $y = x^2$ and $y = x + 2$.

The slices parallel to the y-axis will be perfect squares.

Q: What does the solid look like?



Sometimes you might be given the description of a region in space and you will be asked to find the volume. It is not a bad idea to draw the region in question. By drawing the region, you'll discover such things as how the **cross-sections** are defined as well as what interval the region lies within.

Notice that this drawing is done from a perspective viewpoint. Perspective views are good to illustrate how an object appears three-dimensionally, since viewing the object from above might not reveal as much about its shape.

Determining the area of a soild given its description

Let V be the volume of the solid.

$$V = \int_{-1}^{2} (x+2-x^2)^2 dx$$

$$= \int_{-1}^{2} (x^4 - 2x^3 - 3x^2 + 4x + 4) dx$$

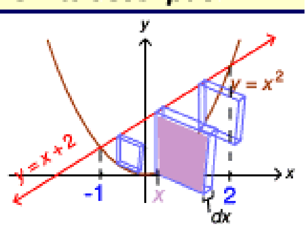
$$= \frac{x^5}{5} - \frac{2x^4}{4} - \frac{x^3}{3} + 2x^2 + 4x \Big|_{-1}^{2}$$

$$= \frac{(2)^5}{5} - \frac{2(2)^4}{4} - \frac{(2)^3}{3} + 2(2)^2 + 4(2) - \frac{(-1)^5}{5} - \frac{2(-1)^4}{4} - \frac{(-1)^3}{3} + 2(-1)^2 + 4(-1)$$

$$= \frac{64}{10} - \quad - \frac{17}{10}$$

$$\cancel{= \frac{47}{10}}$$

$$= \frac{81}{10}$$



Oops: Watch those minus signs.

Since the cross-sections are defined as perfect squares, the first step in finding the volume is to determine the length of the side of the square. The base of each square is equal to the difference of the two functions.

Once you know the length, finding an equation for the area is simple. The cross-sections are defined to be squares. So the area of a cross-section is the square of the length of its base.

Integrating the square of the base along the given interval gives you the volume.

Watch out! When evaluating definite integrals, it is very easy to make algebra mistakes. Watch for them.