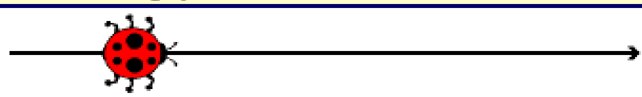
Unit: Parametric Equations and Polar Coordinates **Module:** Understanding Parametric Equations

Introduction to Parametric Equations

key concepts:

- To describe motion in a plane, you need information about the motion in the x-direction and also along the y-direction.
- If f and g are continuous functions of t, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** and t is called the **parameter**.
- If parametric equations describe movement along a path, you can plug in values of the parameter in order to determine the location of the path at every point.
- Parametric equations are a convenient way to describe the path of movement in a plane.

Review of bug questions in one dimension



Given: the position of a bug on a line

$$f(t) = t^2 - t$$

where t is time

Find the velocity of the bug by taking the derivative of $f(t)$:

$$f'(t) = 2t - 1$$

Given a horizontal line, position can be described by a function $f(t)$ where t is time. As the value of t changes, the position changes.

You can differentiate the position function, $f(t)$, to determine the velocity function $f'(t)$.

Equations that describe position in terms of an independent variable are called **parametric equations**. The independent variable is called a **parameter**.

Describing the motion of the bug moving on the xy-plane

If the bug moves around the xy-plane you must have two equations to describe its motion:

1. In the x-direction as a function of time
2. In the y-direction as a function of time

} parametric equations

To describe movement in the plane, you need two functions: one for the x-direction and one for the y-direction. Since there are two directions, you need two parametric equations.

Describing the motion of the bug moving on the xy-plane

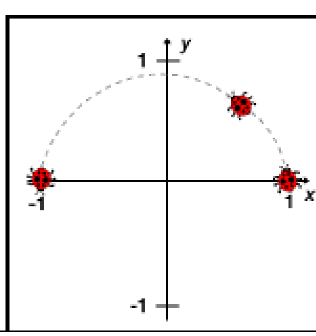
Example: $x(t) = \cos t$ $\quad y(t) = \sin t$ parametric equations

Q: What is the path of the bug?

A: Because you have parametric equations, you can plug in values of t and plot the path of the bug.

At $t = \pi$ minutes

$x(\pi) = \cos \pi = -1 \quad y(\pi) = \sin \pi = 0$



Here are some parametric equations that describe the path of a bug moving in a plane. Notice that the x-direction is denoted $x(t)$ and the y-direction is denoted $y(t)$. This notation lets you know which equation is for what direction.

By taking the parametric equations and plugging in values for t you can plot different positions of the bug.

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The Cycloid

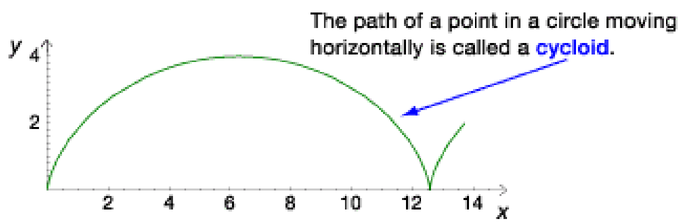
key concepts:

- If f and g are continuous functions of t, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** and t is called the **parameter**.
- A **cycloid** is the curve traced by a point on the circumference of a circle moving along a straight line. The parametric equations that describe a cycloid are $x(\theta) = \theta - \sin(\theta)$ and $y(\theta) = 1 - \cos(\theta)$.

Considering the movement of a point on a bicycle wheel

Q: What is the path of a point on the wheel if the bicycle moves on the ground?

A: The path of the point on a wheel of a bicycle moving on the ground is given by this curve.



Q: How can you determine the equations that describe this curve?

A: Construct parametric equations that describe the x-location and the y-location of each point in the curve.

Did you know? Galileo first called attention to the cycloid and recommended that it be used for arches and bridges. Pascal once spent 8 days solving problems related to the geometry of cycloids. The cycloid is involved in so many interesting and controversial problems that it referred to as "the Helen of geometry."

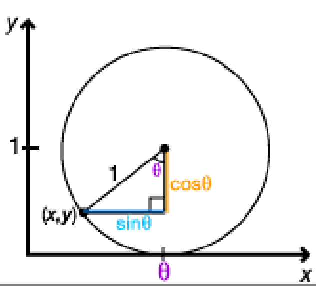
If you look at a wheel as it is moving along a surface it is pretty obvious that the center of the wheel is moving in a straight line.

But what about a point on the wheel itself? It turns out that the point on the wheel moves in a shape known as a **cycloid**. Cycloids look sort of like squashed down circles.

The cycloid is pretty hard to describe in Cartesian form. It is much easier to describe the cycloid with **parametric equations**.

Modeling the movement of a point on a bicycle wheel

Q: What are the equations that describe the location of (x,y)?



- ✔The radius of the circle is 1.
- ✔The Δ is a right triangle.
- ✔The distance the circle has moved is θ.
- ✔The angle the circle has rotated is θ.
- ✔The hypotenuse of the Δ is the radius and has a length of 1.
- ✔The side of the triangle opposite to θ has a length of $\sin\theta$.
- ✔The side of the triangle adjacent to θ has a length of $\cos\theta$.

A: In the x-direction, the point (x,y) is at a distance θ minus $\sin\theta$ from the origin.

$\boxed{x(\theta) = \theta - \sin\theta}$

In the y-direction, the point (x,y) is at a distance 1 minus $\cos\theta$ from the origin.

$\boxed{y(\theta) = 1 - \cos\theta}$

These are the parametric equations that describe a cycloid. The parameter, θ, in these equations can be thought of in two ways: as *time* or as *the angle* that the circle has rotated.

To describe the cycloid with parametric equations, you have to analyze what is happening to the point as the wheel moves. Here the cycloid is described using a circle of radius one.

Start with the circle, the radius, and the point that is moving along the path you want described.

Construct a right triangle from the center of the circle to the point on the circumference. The hypotenuse of the triangle is equal to the radius of the circle. The sides of the triangle can be expressed with trig expressions. The distance the circle has moved in the x-direction is equal to the angle.

Notice that the x-position of the point is given by the distance the circle has traveled minus the sine of the angle. The y-position is given by the height of the center minus the cosine of the angle.

Now you have described the cycloid parametrically.

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Eliminating Parameters

key concepts:

- If f and g are continuous functions of t, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** and t is called the **parameter**.
- One way to graph a parametric equation is to eliminate the parameter through substitution.

Solving for the parameter

Q: How do you plot parametric equations?

Example! Graph $x = t^3 \quad y = t^2$

Technique 1: Plug in values for t and calculate the values of x and y. Plot the resulting (x,y) values.

Technique 2: Throw away the parameter.

A: Eliminating the parameter from parametric equations results in a single equation of the familiar form $y = f(x)$.

Graphing **parametric equations** isn't as intuitive as graphing a single equation in x and y.

You can try to draw the curve by plotting several points. By plugging in values of the **parameter** into both equations you can generate points.

But a quicker way to graph many of these parametric equations is to use algebra to throw away the parameter. Doing so converts the two parametric equations into a single equation in terms of x and y.

Example! Graph $x = t^3 \quad y = t^2$

$x = t^3$

$\sqrt[3]{x} = t$

$y = t^2$

$y = \sqrt[3]{x}^2$ Substitute $\sqrt[3]{x}$ for t

$y = (x^{1/3})^2$ Rewrite the radical as a power.

$y = x^{2/3}$ Multiply exponents.

To eliminate a parameter, just solve for the parameter in one of the parametric equations and substitute that expression into the other parametric equation.

The new equation is much more familiar and easier to graph.

Substitution of part of an equation

Example! $x = e^t$

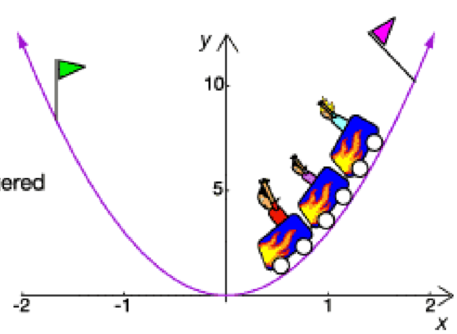
$y = 3e^{2t}$

$y = 3e^{t+t}$ Addition in the exponent means multiply bases.

$y = 3\,e^t \cdot e^t$

$y = 3xx$ Substitute x for e^t.

$y = 3x^2$ Add exponents.



The curve $y = 3x^2$ is concave up, centered at the origin, and narrower than the curve $y = x^2$.

Sometimes it is easier to substitute into the second parametric equation without solving for the parameter. Here solving for t would require introducing logarithms and could get pretty involved.

Instead, you can substitute x for the exponential expression directly.

Now the equation clearly generates a parabola.

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