Introduction to Vector Functions

- Understand the **component functions** associated with **vector functions**.
- Find limits of **vector functions**, applying limit laws such as **L'Hôpital's Rule** in the process.
- Sketch **planar curves** associated with **position vector** functions with values in $\mathbb{R}^2$.
- Sketch **space curves** associated with **position vector** functions with values in $\mathbb{R}^3$.

Components of Vector Functions

Write the component functions for $\vec{\mathbf{r}}(t)$, and find $\lim_{t\to 0} \vec{\mathbf{r}}(t)$.

$$\boxed{\vec{\mathbf{r}}(t) = t^2e^{-t}\vec{\mathbf{i}} + \frac{2t+1}{t^2+1}\vec{\mathbf{j}} + \frac{\sin t}{t}\vec{\mathbf{k}}}$$

$\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$, where

$f(t) = t^2e^{-t}$ ← Component function in the direction of $\vec{\mathbf{i}}$

$g(t) = \frac{2t+1}{t^2+1}$ ← Component function in the direction of $\vec{\mathbf{j}}$

$h(t) = \frac{\sin t}{t}$ ← Component function in the direction of $\vec{\mathbf{k}}$

$$\lim_{t\to 0} \vec{\mathbf{r}}(t) = \left(\lim_{t\to 0} t^2e^{-t}\right)\vec{\mathbf{i}} + \left(\lim_{t\to 0} \frac{2t+1}{t^2+1}\right)\vec{\mathbf{j}} + \left(\lim_{t\to 0} \frac{\sin t}{t}\right)\vec{\mathbf{k}}$$
$$= 0\vec{\mathbf{i}} + 1\vec{\mathbf{j}} + 1\vec{\mathbf{k}}$$

Finding limits of vector functions is as easy as finding the limits of each of the component functions!

L'Hôpital's Rule: $\lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{\cos t}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$

A **vector function** is a function whose domain is a set of real numbers and whose range is a set of vectors.

For a vector function $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$, $f(t)$, $g(t)$, and $h(t)$ are real-valued functions called the **component functions** of $\vec{\mathbf{r}}(t)$, and $\vec{\mathbf{r}}(t)$ may also be written as $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$.

Finding the limit of a vector function $\vec{\mathbf{r}}(t)$ as t approaches 0 is just a matter of finding the limits of each of the component functions as t approaches 0.

Recall **L'Hôpital's Rule**, which applies to the third component function because the numerator and the denominator are differentiable on an open interval containing 0 (except possibly at 0), and the limit of the quotient as t approaches 0 is the indeterminate form, $\frac{0}{0}$. L'Hôpital's Rule says that the limit of the quotient is equal to the limit of the quotient of the derivatives of the numerator and denominator, as long as the limit exists.

Sketching Curves Defined by Vector Functions

Sketch the planar curve represented by the vector function.

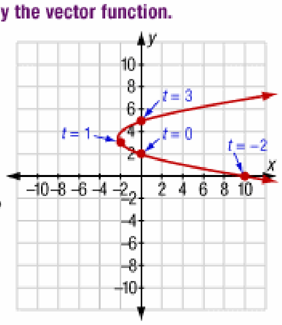
$$\boxed{\vec{\mathbf{r}}(t) = \langle t^2 - 3t,\ t + 2\rangle}$$

$\vec{\mathbf{r}}(0) = \langle 0^2 - 3\cdot 0, 0 + 2\rangle = \langle 0, 2\rangle$

$\vec{\mathbf{r}}(1) = \langle 1^2 - 3\cdot 1, 1 + 2\rangle = \langle -2, 3\rangle$

$\vec{\mathbf{r}}(3) = \langle 3^2 - 3\cdot 3, 3 + 2\rangle = \langle 0, 5\rangle$

$\vec{\mathbf{r}}(-2) = \langle -2^2 - 3\cdot(-2), -2 + 2\rangle = \langle 10, 0\rangle$

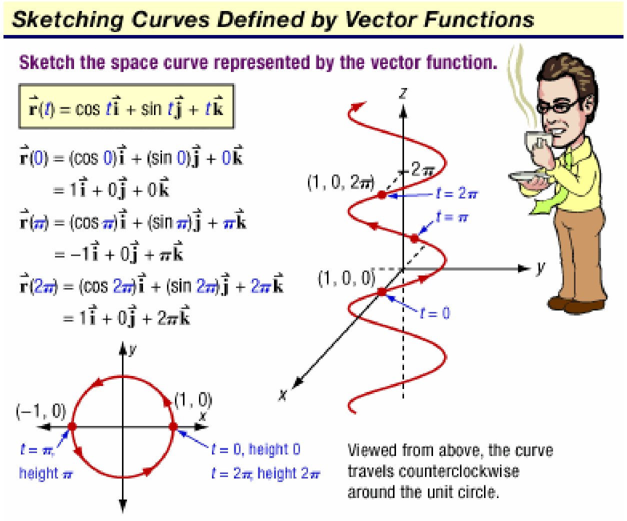


A **planar curve** is the set of points (x, y) that satisfy parametric equations $x = f(t)$, $y = g(t)$, where $f(t)$ and $g(t)$ are continuous. It may be described by a 2-dimensional vector function, also referred to as the **position vector**,

$\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}}$.

To sketch a planar curve, evaluate the vector function at various t values, and then plot the resulting points (x, y), where $x = f(t)$ and $y = g(t)$, on the coordinate axes.

The curve can be thought of as moving in the direction that corresponds to increasing values of t.



The set of points (x, y, z) satisfying $x = f(t)$, $y = g(t)$, and $z = h(t)$, for $f(t)$, $g(t)$, and $h(t)$ continuous, is called a **space curve**. It may be described by a 3-dimensional vector function, also referred to as the **position vector**,

$$\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}.$$

To sketch a space curve, proceed in exactly the same way, by evaluating the vector function at various t values and then plotting the resulting points on a 3-dimensional coordinate system.

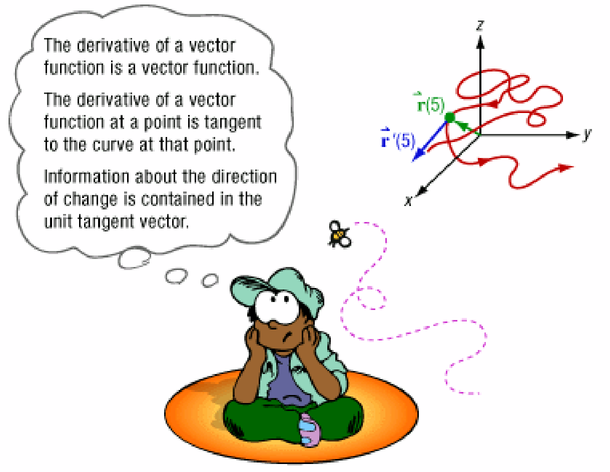
For this vector function, the x and y values travel counterclockwise around the unit circle, and the z value (the height) varies with t.

The spiral formed is called a helix.

Derivatives of Vector Functions

- Understand the definition of the **derivative of a vector function** in terms of limits.
- Find the derivative of a vector function by taking the derivative of its components.
- Find the **tangent vector** and the **unit tangent vector** at a point.
- Sketch a planar curve described by a vector function, and find and sketch its position and **tangent vectors** at a point.

Understanding Derivatives of Vector Functions



A position vector function traces out a curve. The points on the curve correspond to the tips of the vectors obtained when the position vector function is evaluated at various values of *t*. These position vectors should be thought of as emanating from the origin.

The **derivative of a vector function** is itself a vector function. For each time *t*, the derivative of the position vector function at *t* is a vector that is tangent to the curve at *t*.

Given a vector function $\vec{r}(t)$ with values in $\mathbb{R}^2$ or $\mathbb{R}^3$, the **derivative of $\vec{r}(t)$** is

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}.$$

The definition of the **derivative of a vector function** is the same as the standard definition of a derivative. Take a small step *h* in the *t* direction, and look at the difference of the two vectors obtained at *t* and *t* + *h*, all divided by *h*. Then let *h* go to 0. The result is an instantaneous rate of change at *t*, where rate of change is now expressed as a vector.

Theorem: If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$, then $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j}$.

If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, then $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$.

Luckily, this definition of the derivative of a vector function leads to an easy way to compute the derivative. Just take the derivative of each of the component functions of the vector function $\vec{\mathbf{r}}(t)$, and then use these derivatives as the new components of $\vec{\mathbf{r}}'(t)$.

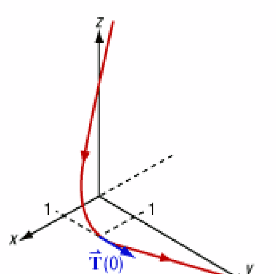
Finding Derivatives of Vector Functions

Find the derivative of $\vec{r}(t) = (\cos 2t)\vec{i} + (1+t)e^{3t}\vec{j} + 4t^2\vec{k}$ and find the unit tangent vector at the point where $t = 0$.

$$\vec{r}\,'(t) = (-2\sin 2t)\vec{i} + [(1+t)3e^{3t} + e^{3t}]\vec{j} + 8t\vec{k}$$
$$= (-2\sin 2t)\vec{i} + (4+3t)e^{3t}\vec{j} + 8t\vec{k}$$

$$\vec{r}\,'(0) = (-2\sin(2\cdot 0))\vec{i} + (4+3\cdot 0)e^{3\cdot 0}\vec{j} + 8\cdot 0\vec{k}$$
$$= 0\vec{i} + 4\vec{j} + 0\vec{k}$$
$$= 4\vec{j}$$

$$\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{0\vec{i} + 4\vec{j} + 0\vec{k}}{\sqrt{0^2+4^2+0^2}}$$
$$= 0\vec{i} + 1\vec{j} + 0\vec{k} = \vec{j}$$



$\vec{r}(t) = (\cos 2t)\vec{i} + (1+t)e^{3t}\vec{j} + 4t^2\vec{k}$

This vector function describes a complicated curve in $\mathbb{R}^3$. The **unit tangent vector** is a vector that has the same direction as the tangent vector, but is scaled to have length equal to one.

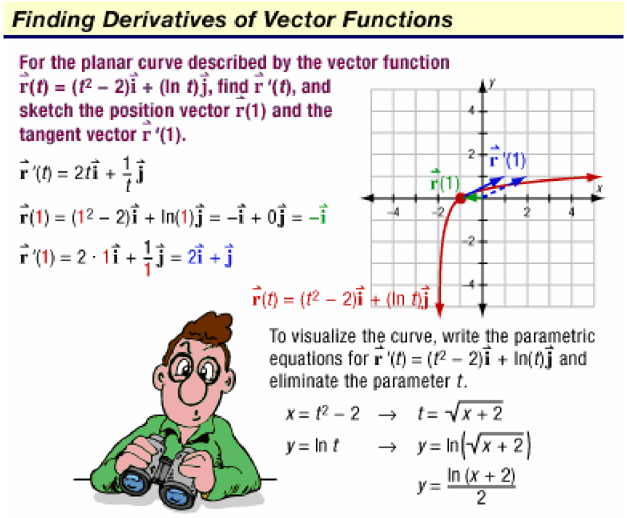
To find the unit tangent vector at $t = 0$, first compute the **tangent vector**, $\vec{\mathbf{r}}'(t)$. Then evaluate the tangent vector at $t = 0$. The result is a vector.

Next, scale the vector $\vec{\mathbf{r}}'(0)$ by dividing each of its components by the magnitude of $\vec{\mathbf{r}}'(0)$.

Given a vector function $\vec{r}(t)$ with values in $\mathbb{R}^2$ or $\mathbb{R}^3$, the **unit tangent vector** at *t* is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}.$$

More generally, the **unit tangent vector** at *t* is the tangent vector at *t* divided by its magnitude.



The vector function in this example describes a planar curve. The first task is to find the tangent vector $\vec{\mathbf{r}}'(t)$. Just compute the derivatives of each of the components of $\vec{\mathbf{r}}(t)$ and form the derivative vector function out of the new components.

To sketch the position vector at $t = 1$, evaluate $\vec{\mathbf{r}}(t)$ at $t = 1$, and sketch the vector obtained, starting at the origin.

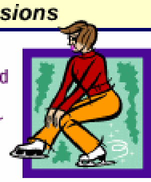
To sketch the tangent vector $\vec{\mathbf{r}}'(t)$ at $t = 1$, evaluate $\vec{\mathbf{r}}'(t)$ at $t = 1$, and sketch the vector obtained, starting at $\vec{\mathbf{r}}(1)$.

Vector Functions: Velocity and Acceleration

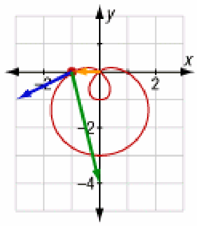
- Use vector functions to represent **position**, **velocity**, and **acceleration** in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Find the **velocity** and **acceleration** vectors for an object at a particular time, given a vector function for its **position**.
- Graph vectors for the **position**, and the direction of the **velocity** and **acceleration** of an object at a particular time, given a vector function for its **position**.
- Find the velocity and acceleration vector functions for an object, given a vector function for its **position**.
- Find the **speed** of an object as a function of time, given a vector function for its **position**.

Velocity and Acceleration in Two Dimensions

Laura is ice skating and traces out a path on the ice given by the vector function shown, where t is measured in seconds. Find Laura's velocity and acceleration after 5 seconds. Draw three vectors on the curve to show her position, and the **direction** of both her velocity and acceleration vectors when $t = 5$ seconds.



$$\vec{\mathbf{r}}(t) = \langle \cos \pi t - \sin 2\pi t,\ \sin \pi t - 2\sin^2 \pi t \rangle$$



$$\vec{\mathbf{r}}(5) = \langle \cos 5\pi - \sin 10\pi,\ \sin 5\pi - 2\sin^2 5\pi \rangle = \langle -1, 0 \rangle$$

$$\vec{\mathbf{v}}(5) = \langle -\pi \sin 5\pi - 2\pi \cos 10\pi,\ \pi \cos 5\pi - 2\pi \sin 10\pi \rangle = \langle -2\pi, -\pi \rangle = \pi\langle -2, -1 \rangle$$

$$\vec{\mathbf{a}}(5) = \langle -\pi^2 \cos 5\pi + 4\pi^2 \sin 10\pi,\ -\pi^2 \sin 5\pi - 4\pi^2 \cos 10\pi \rangle = \langle \pi^2, -4\pi^2 \rangle = \pi^2\langle 1, -4 \rangle$$

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = \langle -\pi \sin \pi t - 2\pi \cos 2\pi t,\ \pi \cos \pi t - 4\pi \sin \pi t \cos \pi t \rangle = \langle -\pi \sin \pi t - 2\pi \cos 2\pi t,\ \pi \cos \pi t - 2\pi \sin 2\pi t \rangle$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \langle -\pi^2 \cos \pi t + 4\pi^2 \sin 2\pi t,\ -\pi^2 \sin \pi t - 4\pi^2 \cos 2\pi t \rangle$$

The position vector function outputs Laura's **position** on the ice in two dimensions for each time input. To find Laura's position after 5 seconds, substitute 5 into the position vector function.

The velocity vector function outputs Laura's **velocity**, which contains information about how fast she is going and in what direction, for each time input. To find Laura's velocity after 5 seconds, first take the derivative of her position vector function. Then, substitute 5 into the resulting velocity vector function.

The acceleration vector function outputs Laura's **acceleration**, which contains information about the rate and direction of change of her velocity, for each time input. To find Laura's acceleration after 5 seconds, first take the derivative of her velocity vector function. Then, substitute 5 into the resulting acceleration vector function.

Position: $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}}$ **Speed:** $|\vec{\mathbf{v}}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2}$

Velocity: $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = f'(t)\vec{\mathbf{i}} + g'(t)\vec{\mathbf{j}}$ **Acceleration:** $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = f''(t)\vec{\mathbf{i}} + g''(t)\vec{\mathbf{j}}$

The formulas for the position, velocity, and acceleration vector functions in two dimensions are shown here, along with the formula for **speed**, which is a function whose value at t is the magnitude of the velocity function at t.

Velocity, Acceleration, and Speed in Three Dimensions

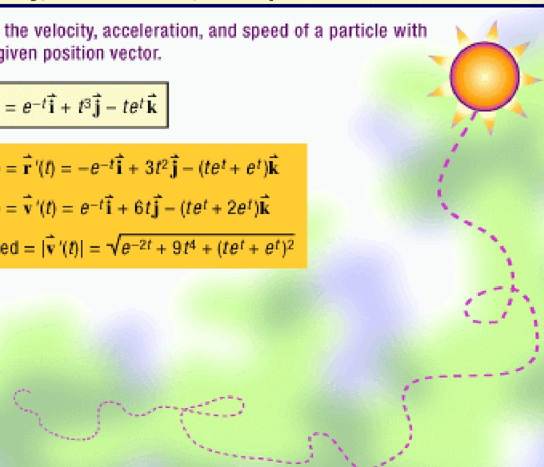
Find the velocity, acceleration, and speed of a particle with the given position vector.

$$\vec{\mathbf{r}}(t) = e^{-t}\vec{\mathbf{i}} + t^3\vec{\mathbf{j}} - te^t\vec{\mathbf{k}}$$

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = -e^{-t}\vec{\mathbf{i}} + 3t^2\vec{\mathbf{j}} - (te^t + e^t)\vec{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = e^{-t}\vec{\mathbf{i}} + 6t\vec{\mathbf{j}} - (te^t + 2e^t)\vec{\mathbf{k}}$$

$$\text{speed} = |\vec{\mathbf{v}}\,'(t)| = \sqrt{e^{-2t} + 9t^4 + (te^t + e^t)^2}$$



In this problem, a particular time is not given. The goal is to find the velocity vector function, the acceleration vector function, and the function for the speed of the particle, which is moving in three dimensions.

To find the vector function for the velocity of the particle, take the derivative of the position vector function. To find the vector function for the acceleration of the particle, take the derivative of the velocity vector function.

To find the function for the speed of the particle, take the magnitude of the velocity vector function. Recall that the magnitude of a vector is just its length, and can be obtained by taking the square root of the sum of the squares of its components.

Position: $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$ **Speed:** $|\vec{\mathbf{v}}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$

Velocity: $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = f'(t)\vec{\mathbf{i}} + g'(t)\vec{\mathbf{j}} + h'(t)\vec{\mathbf{k}}$ **Acceleration:** $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = f''(t)\vec{\mathbf{i}} + g''(t)\vec{\mathbf{j}} + h''(t)\vec{\mathbf{k}}$

The formulas for the position, velocity, and acceleration vector functions in three dimensions are shown here, along with the formula for speed, which is a function whose value at t is the magnitude of the velocity function at t.