Introduction to Infinite Series

key concepts:

- **Binary operations** combine two values to yield a single result.
- You can add as many numbers as you want as long as you only have finitely many of them. Since addition is for finitely many numbers, you will encounter difficulties if you try to add infinitely many terms.
- An **infinite series** is the sum of an infinite collection of terms.

Returning to addition

Consider your first addition problem.

binary operation $1+1=2$

$5+3=8$

Generalize addition. $1+1=2$

$1+1+1=3$

To add three numbers, add the first two, and then add the result to the third.

Repeat this for more numbers, as long as there are only finitely many of them.

Question What if you have infinitely many terms?

$1+1+1+1+\ldots=?$

The definition of addition is a problem, because addition is only defined for finitely many terms. Answer

If you take a careful look at addition, you will notice that addition is defined as the combination of two numbers. This is called a **binary operation**.

The only reason you can add multiple numbers together is because of the associative property.

You can add as many numbers together as you like and get the same result no matter which order you combine them as long as you have finitely many numbers.

If you have infinitely many terms, then addition becomes a little trickier. It turns out that sometimes adding infinitely many terms different ways will produce different answers. The sum of an infinite collection of terms is called an **infinite series**.

Infinite series

binary operation $1+1=2$

Example! Consider the following series:

$$1+-1+\ 1+-1+\ 1+\ldots = ?$$

$$0\ +\ 0\ +\ 0+\ldots = 0$$

$$1+\ 0\ +\ 0\ +\ 0+\ldots = 1$$

But zero does not equal one! This is a problem, and it is the inspiration for studying how to add things up infinitely often.

Let's consider the infinite series that consists of adding positive one to negative one to positive one to negative one, etc.

Since you can group each pair together to add zero, you could argue that the sum is zero.

But you could also show that the series would equal one by ignoring the first term and grouping the remaining terms the same way.

Zero does not equal one! There must be something more going on when dealing with infinite series.

Introduction to Infinite Series

Writing down infinitely many things

Suppose you want to add up the following:

$$a_1 + a_2 + a_3 + a_4 + \ldots$$

Instead write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$$

Example!

$$\sum_{n=1}^{3} \frac{n}{n+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1}$$
$$= \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$$

When n reaches four, you stop, because the index only goes to three.

When dealing with infinite series, there are some notation conventions you need to know.

Instead of writing each term of the infinite series, you can use a capital sigma to denote "sum" and follow it with the general term of a sequence. The sigma tells you to sum up the sequence. The line under the sigma tells you what the index is and where to start and the value on top tells you where to finish.

Here is a finite example. The line underneath the sigma is $n = 1$. That means n is the index and the first term of the sum is the term for $n = 1$. The line above the sigma is a three, which tells you that you only add terms until you reach the third.

The sequence tells you what the individual terms are. Just plug the index into the general term of the sequence for each term you need to add.

Writing down infinitely many things

Some more examples of infinite series . . .

Example!

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$$

Read this as "the summation as n goes from one to infinity of one over n."

Example!

$$\sum_{t=1}^{\infty} \frac{1}{t^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

You can use any letter, it does not have to be n.

Example!

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \ldots$$

The sigma notation is used to write a long sum compactly.

Here are some more examples of series.

In this example the index is n. The starting point is $n = 1$. The ending point is infinity.

When reading the statement, the sigma is read as "the summation". Then you state the index, n. Then you say what n is allowed to be.

This statement is read as "the summation as t goes from one to infinity of one over t-squared." Notice that you can use whatever you want as your index.

This statement is read as "the summation as k goes from one to infinity of the quotient of k and $k + 1$.

You don't have to start the index equal to one. If the lower line had read $k = 2$ then the summation would have gone from two to infinity.

Summation of Infinite Series

key concepts:

- Given an **infinite series** $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ and the sequence $\{s_m\}$ of **partial sums**, then $\sum_{n=1}^{\infty} a_n$ **converges** if $\lim_{m \to \infty} s_m = S$. In this case $\sum_{n=1}^{\infty} a_n = S$.

  The series $\sum_{n=1}^{\infty} a_n$ **diverges** if $\lim_{m \to \infty} s_m$ does not exist.

- You will not be able to determine the sum of most series. However, you can determine whether the series converges or diverges.

Partial sums of an infinite series

Consider the general infinite series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$

Question? How do you make sense of adding up infinitely many terms?

Answer: Solve an easier problem instead and use the result as a starting point for the original question.

Start by adding a few terms at a time.

First sum: $s_1 = a_1 + a_2$

Second sum: $s_2 = a_1 + a_2 + a_3$

Third sum: $s_3 = a_1 + a_2 + a_3 + a_4$

$\vdots$

General term: $s_m = a_1 + a_2 + a_3 + a_4 + \ldots + \boxed{a_{m+1}}$

The last term of each sum has an index which is one larger than the index of the sum.

To add up an **infinite series**, start by looking at the sequence of **partial sums**. A partial sum is the value you get when you examine only a finite number of terms of the series.

Notice that the first partial sum requires two terms of the sequence of the series to evaluate. In general, the *n*th partial sum will require $n + 1$ terms to evaluate.

You can compute the general term, because you are only adding finitely many numbers.

The sums $s_1, s_2, s_3 \ldots$ form a sequence of numbers, $\{s_m\}$.

$\{s_m\}$ is the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$

Question? What is the limit of the sequence $\{s_m\}$?

Answer: If the limit of the sequence $\{s_m\}$ of the partial sum exists, then the terms of the infinite series can be summed up. The limit of the sequence equals the sum. Thus the infinite series converges.

If the limit does not exist, then the series diverges.

Putting each partial sum in an ordered list creates the sequence of partial sums. Notice that the sequence of partial sums tells what the series would equal if it was a finite series.

It is possible to determine what value the sequence of partial sums is approaching by taking a limit.

If the limit of a sequence of partial sums approaches some value, then the sequence has a limit. If the sequence has a limit, then the series **converges** to the same limit. This is how you add infinite series.

If the sequence of partial sums does not have a limit, then the series **diverges**.

Summation of Infinite Series

Summation of a well-known series

Example! Look at the following series again:

$$\sum_{n=1}^{\infty}(-1)^{n+1}=1-1+1-1+1\ldots$$

Now look at the associated sequence of partial sums.

odd $s_1=1-1=0$
even $s_2=1-1+1=1$
odd $s_3=1-1+1-1=0$
even $s_4=1-1+1-1+1=1$

Question ? Is this sequence approaching a particular value?

Answer No, the $\lim_{m\to\infty} s_m$ does not exist. Therefore the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}=1-1+1-1+1\ldots$ ← Diverges

Consider this series, sometimes called the devil's series because it perplexed mathematicians for so long.

You saw before how you could argue that the sum of the series was one or zero depending on how you grouped the terms.

Now instead of adding terms, look at the sequence of partial sums.

The sequence of partial sums changes at each term, switching between one to zero.

The sequence does not approach a single value, therefore it does not have a limit. That means the series diverges.

A practical consideration

To determine if an infinite series converges, take the limit of the sequence of partial sums.

$$s_1=a_1+a_2$$
$$s_2=a_1+a_2+a_3$$
$$s_3=a_1+a_2+a_3+a_4$$
$$\vdots$$
$$s_m=a_1+a_2+a_3+a_4+\ldots+a_{m+1}$$

Question ? How can you find the sequence of partial sums? And how do you find its limit?

Answer In most cases, it is too complicated to do either. Therefore, you cannot determine what the infinite series sums to.

You can only determine whether the series sums to a number or not.

Although it seems like you could use this approach to find the value of any series, in truth it is kind of hard to add an infinite number of terms together to find out if a series is approaching some value.

A lot of different series problems out there cannot be evaluated directly. But it isn't always necessary to find the summation of the series. Sometimes just knowing that the series converges is information enough. In the following tutorials you will learn about some common series, as well as some ways to test series for convergence.

Geometric Series

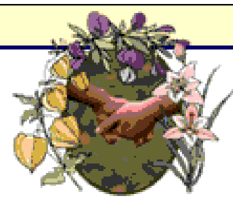
key concepts:

- Given an **infinite series** $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ and the sequence $\{s_m\}$ of **partial sums**, then $\sum_{n=1}^{\infty} a_n$ **converges** if $\lim_{m\to\infty} s_m = S$. In this case $\sum_{n=1}^{\infty} a_n = S$.

  The series $\sum_{n=1}^{\infty} a_n$ **diverges** if $\lim_{m\to\infty} s_m$ does not exist.

- In a **geometric series**, $\sum_{n=1}^{\infty} ar^n = ar^1 + ar^2 + ar^3 \ldots$, if $|r| < 1$ then the series converges and $\sum_{n=1}^{\infty} ar^n = \dfrac{ar}{1-r}$. If $|r| \geq 1$ then the series diverges.

The friendly geometric series



Consider the geometric series

$$\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n = \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \ldots$$

Q? Does this series converge, and if so, what is its sum?

The first general type of **infinite series** you will learn about is called the **geometric series**. The geometric series is a friendly series, for reasons you will discover below.

A geometric series occurs when you have a number raised to the power of the index of your series.

To find the sum of this geometric series, start by looking at the terms.

Let s_m be the sum of the first m terms.

$$\frac{10}{10} s_m = \frac{1}{10} + \cancel{\left(\frac{1}{10}\right)^2} + \cancel{\left(\frac{1}{10}\right)^3} + \ldots + \cancel{\left(\frac{1}{10}\right)^m}$$

$$-\left(\frac{1}{10} s_m\right) = -\left[\cancel{\left(\frac{1}{10}\right)^2} + \cancel{\left(\frac{1}{10}\right)^3} + \ldots + \cancel{\left(\frac{1}{10}\right)^m} + \left(\frac{1}{10}\right)^{m+1}\right]$$

Multiply by $\frac{1}{10}$.

$$\frac{9}{10} s_m = \frac{1}{10} - \left(\frac{1}{10}\right)^{m+1}$$

All the other terms cancel.

$$s_m = \frac{10}{9}\left[\frac{1}{10} - \left(\frac{1}{10}\right)^{m+1}\right]$$

Multiply both sides by $\frac{10}{9}$.

$$s_m = \frac{1}{9} - \frac{10}{9}\left(\frac{1}{10}\right)^{m+1}$$

So $\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$ converges if $\lim_{m\to\infty} s_m$ exists.

Consider the **partial sums** of the geometric series.

If you subtract one-tenth of the partial sum from each term of the sequence, then you will notice that you can cancel out all of the terms except the first and last.

The resulting sum equals nine-tenths of the partial sum for any given term.

Simplify by isolating the partial sum.

Now you have an equation that tells you what the partial sum equals for any given index m.

If the limit of this expression exists as the index approaches infinity, then the sequence of partial sums has a limit and therefore the series **converges**. If the limit does not exist, then the series **diverges**.

Geometric Series

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty}\left[\frac{1}{9}-\frac{10}{9}\left(\frac{1}{10}\right)^{m+1}\right]$$
$$= \lim_{m\to\infty}\frac{1}{9} - \lim_{m\to\infty}\frac{10}{9}\left(\frac{1}{10}\right)^{m+1}$$
$$= \frac{1}{9} - 0$$
$$= \boxed{\frac{1}{9}}$$

As you raise $\frac{1}{10}$ to higher powers, it approaches zero.

A! The limit of the sequence of partial sums is $\boxed{\frac{1}{9}}$.
So the infinite series $\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$ converges to $\boxed{\frac{1}{9}}$.

By taking the limit as the index approaches infinity you can determine if the sequence of partial sums has a limit.

Notice that the sequence has a limit. The sequence approaches one-ninth.

This means that the series converges to that same value.

Generalizing the geometric series

The general form of the geometric series

is $\sum_{n=1}^{\infty} r^n = r^1 + r^2 + r^3 + \ldots$, where r is a constant.

If $|r| < 1$, then

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

If $|r| \geq 1$, then $\sum_{n=1}^{\infty} r^n$ diverges.

The number being raised to the power of the index is called the ratio. For geometric series, if the absolute value of the ratio is less than one, then the series converges. If the absolute value of the ratio is greater than or equal to one, then the series diverges.

Example!

If $r = 3$,

$$\sum_{n=1}^{\infty} 3^n = 3^1 + 3^2 + 3^3 + \ldots$$
$$= 3 + 9 + 27 + \ldots$$

This sum will not converge to a number.

Consider the example where the ratio equals three.

Three is greater than one, so the series should diverge. If you look at the partial sums you will notice that the terms of the sequence are increasing very quickly. The sequence of partial sums has no limit, so the series must diverge.

More general forms

Some infinite series start at $n = 0$ and have a constant a.

$$\sum_{n=0}^{\infty} ar^n = ar^0 + ar^1 + ar^2 + ar^3 + \ldots$$

If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

The general form can also start with $n = 1$.

$$\sum_{n=1}^{\infty} ar^n = ar^1 + ar^2 + ar^3 + \ldots$$

If $|r| < 1$, then

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$$

Be very careful when working with geometric series. If the index starts at zero instead of one, the formula for the summation of the series is a little different.

Also notice that if you multiply each term of the series by a constant the result of the series is equal to the result given by that ratio, times the constant.

It is sort of like you can factor out the constant and multiply it by the value of the series at the end. In fact, that is exactly what you are doing.

Unit: Sequences and Series **Module:** Infinite Series

Telescoping Series

key concepts:

- A **telescoping series** is a series of the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$.
- To find the sum of a telescoping series, use a partial fraction decomposition to collapse the series down to a closed form and then take the limit of that expression.

Partial fractions of a series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Q? Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Use partial fraction decomposition.

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

For $n = 1$ For $n = 2$ For $n = 3$

Write the expression for the sum of the first m terms.

$$s_m = \left(1 - \cancel{\frac{1}{2}}\right) \cancel{+} \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) \cancel{+} \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \cancel{+ \ldots +} \left(\cancel{\frac{1}{m}} - \frac{1}{m+1}\right)$$

Cancel terms.

$$s_m = 1 - \frac{1}{m+1}$$

Write the closed form.

Consider this infinite series. At first glance, it isn't obvious how you might find the sum of this series.

But you can rewrite the terms by using a **partial fraction decomposition**.

Notice that by dividing the expression into partial fractions you can visualize the underlying pattern of the series better.

The important pattern of this series is that a piece of each term is cancelled by the following term. Writing out a partial sum illustrates how only the first and last terms impact the series. A series that behaves like this is called a **telescoping series**.

Partial fractions of a series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Q? Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \boxed{1}$$

The limit of the partial sum $s_m = 1 - \frac{1}{m+1}$

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right)$$

$$= \boxed{1}$$

As m gets larger, the fraction approaches zero.

The limit of the partial sums is one.
The series converges.

By finding the partial sum and canceling those terms you can, you produce an expression for the mth partial sum that is much easier to evaluate.

Now you can take the limit of the sequence of the partial sums to find out if the series converges or not.

Notice that the limit exists and equals one. That means the series converges to one.

So if you can express a series in telescoping form, then you have a good chance of finding its limit.