Unit: Techniques of Integration **Module:** Integration by Partial Fractions with Repeated Factors

Repeated Linear Factors – Part One

key concepts:

- Use **polynomial long division** when the degree of the numerator of a rational function is higher than the degree of the denominator.
- The **partial fractions** technique must be modified when factors of the denominator are repeated.

Using polynomial long division

Consider $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$ (1)

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1 \overline{)\, x^4+0x^3-2x^2+4x+1} \\ -\left(x^4-x^3-x^2+x\right) \\ \hline x^3-x^2+3x+1 \\ -\left(x^3-x^2-x+1\right) \\ \hline 4x \end{array}$$

Include placeholders for missing powers.

remainder → $4x$

So $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$

The new integral is $\int\left(x+1+\frac{4x}{x^3-x^2-x+1}\right)dx$ (2), which is less complicated than the original expression.

Notice that the degree of the numerator of this rational function is greater than the degree of the denominator. Since the integral can't be evaluated with u-substitution, start by trying to simplify the integral with **polynomial long division**.

Polynomial long division is similar to arithmetic long division. Choose each term by determining what you would multiply the divisor by in order to remove the leading term of the dividend. Just like in arithmetic long division, repeat the process until the divisor can no longer divide the dividend.

Make sure to include the remainder divided by the divisor in your result. Without this piece, your division problem is wrong.

Looking for a pattern

(1) Consider $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

(2) $\int\left(x+1+\frac{4x}{x^3-x^2-x+1}\right)dx$

Factor the denominator.

$$\begin{aligned}\frac{4x}{x^3-x^2-x+1} &= \frac{4x}{x^2(x-1)-x+1}\\ &= \frac{4x}{x^2(x-1)-(x-1)}\\ &= \frac{4x}{(x-1)(x^2-1)}\\ &= \frac{4x}{(x-1)(x-1)(x+1)}\\ &= \frac{4x}{(x-1)^2(x+1)}\end{aligned}$$

The new expression of the integral is

(3) $\int\left(x+1+\frac{4x}{(x-1)^2(x+1)}\right)dx$

You still have the first two terms, which you know how to integrate.

Now use the technique of partial fractions to integrate the fractional term.

Q How do you work with a fraction that has a quantity squared in the denominator?

A Modify the partial fraction technique.

Notice that in the new integral, the first two terms are easy to integrate. The final term is a little more complicated, but a **partial fraction decomposition** should work.

However, notice that when factored the denominator has a term repeated. This factor might complicate the partial fractions technique.

Repeated Linear Factors – Part Two

key concepts:

- Use **polynomial long division** when the degree of the numerator of a rational function is higher than the degree of the denominator.
- The **partial fractions** technique must be modified when factors of the denominator are repeated.
- To modify the partial fractions technique to work with repeated factors, each power of a factor that is repeated in the denominator must get its own partial fraction.

Divide and conquer

Take another look at $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$.

Since the degree of the numerator is higher than the degree of the denominator, you can use polynomial long division.

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left(x + 1 + \frac{4x}{(x-1)^2 (x+1)} \right) dx$$

Now you have a fraction that you could try to break up into partial fractions.

Remember Use polynomial long division when the degree of the numerator is higher than the degree of the denominator.

When the numerator of the rational function is of greater order than the denominator, you can simplify the expression by using **polynomial long division**.

Remember, the remainder goes over the original denominator and adds to the result of the division. It is also a good idea to factor the denominator.

Taking care of the square

$$\int \left(x + 1 + \frac{4x}{(x-1)^2 (x+1)} \right) dx$$

Q How would you handle the squared factor?

A Don't forget the square! Include a fraction whose denominator has the squared factor.

You want to break up:

$$\frac{4x}{(x-1)^2 (x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} \quad \text{Establish an identity.}$$

$$= \frac{A(x-1)(x+1)}{(x-1)^2 (x+1)} + \frac{B(x+1)}{(x-1)^2 (x+1)} + \frac{C(x-1)^2}{(x-1)^2 (x+1)} \quad \text{Find a common denominator.}$$

$$= \frac{A\left(x^2 - 1\right) + B(x+1) + C\left(x^2 - 2x + 1\right)}{(x-1)^2 (x+1)}$$

$$= \frac{Ax^2 - A + Bx + B + Cx^2 - 2Cx + C}{(x-1)^2 (x+1)} \quad \text{Distribute.}$$

$$= \frac{(A+C)x^2 + (B-2C)x - A + B + C}{(x-1)^2 (x+1)} \quad \text{Regroup with respect to powers of } x.$$

If one of the factors of the denominator of the rational function is repeated then the technique of **partial fractions** must be modified.

Each power of the repeated factor gets its own partial fraction. If the term were raised to the third power, then there would be a partial fraction for the first power, another for the second power, and another for the third.

Once you put in the additional terms for any repeated factors you can work out the partial fraction decomposition normally. Find the common denominator and distribute.

Regroup the expressions with respect to the powers of x so you can solve for the unknown constants.

Calculus Lecture Notes

thinkwell ©2001

Unit: Techniques of Integration **Module:** Integration by Partial Fractions with Repeated Factors

Repeated Linear Factors – Part Two

Taking care of the square

$$\int \left(x+1+\frac{4x}{(x-1)^2(x+1)} \right) dx$$

$$\frac{0x^2+4x+0}{(x-1)^2(x+1)}=\frac{(A+C)x^2+(B-2C)x-A+B+C}{(x-1)^2(x+1)}$$

You have an identity, so you can deduce the following:

$A+C=0$ $\quad B-2C=4$ $\quad -A+B+C=0$

$A=-C$

Solve for A.

$-(-C)+B+C=0$

$C+B+C=0$

$B+2C=0$

$B=-2C$

Substitute the value of A.

Solve for B.

$B+B=4$

$2B=4$

$B=2$

Substitute the value of $-2C$ into the equation.

$A=-(-1)$ $\quad 2=-2C$

$A=1$ $\quad B=2$ $\quad C=-1$

Once you have grouped the constants with respect to the powers of x you can use whatever technique you want to solve for them.

Here we start by solving the first equation for A.

Next we substitute that value into the third equation and express B in terms of C.

Now we can find the numerical value for B.

With that information, finding the numerical values of A and C is easy.

Partial fractions

$$\int \left(x+1+\frac{4x}{(x-1)^2(x+1)} \right) dx$$

$$\frac{4x}{(x-1)^2(x+1)}=\frac{1}{(x-1)}+\frac{2}{(x-1)^2}+\frac{-1}{(x+1)}$$

It's an IDENTITY

So the integral becomes

$$\int \left(x+1+\frac{4x}{(x-1)^2(x+1)} \right) dx=\int \left(x+1+\frac{1}{(x-1)}+\frac{2}{(x-1)^2}-\frac{1}{(x+1)} \right) dx$$

$$\boxed{=\frac{x^2}{2}+x+\ln|x-1|-\frac{2}{(x-1)}-\ln|x+1|+C}$$

$u=x-1$

$du=dx$

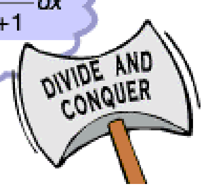
$$\int \frac{1}{(x-1)^2}dx=\int \frac{1}{u^2}du$$

$$=-u^{-1}+C$$

$$=\frac{-1}{u}+C$$

$$=\frac{-1}{x-1}+C$$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$$



Armed with the partial fraction decomposition, the integral is much easier to solve.

Make the substitution using the identity you just discovered above.

Each of these terms can be integrated with the power rule, although some of them require a simple u-substitution.

So remember, when dealing with complicated rational expressions, start by seeing if you need to divide it out.

When you encounter repeated factors in the denominator, remember to repeat them in the partial fraction decomposition.

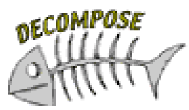
When working math problems, it really helps to write a little more carefully than you would normally do. If you can prevent an error by writing larger, do so!

Distinct and Repeated Quadratic Factors

key concepts:

- When decomposing an expression with repeated factors, each power of the repeated factor gets its own unknown.
- If the factor of an expression is an irreducible quadratic, then the factor gets two unknowns when it is decomposed, one multiplied by x and the other a constant. These factors use the form $Cx + D$ in their numerators.

Repeated linear functions



Suppose you wanted to decompose the following rational expression, which has more than one repeated linear factor:

$$\frac{1}{(x-1)^2(x+2)^3(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3} + \frac{F}{(x-4)}$$

$$\frac{1}{(x-1)^2(x+2)^3(x-4)} = \frac{2/243}{(x-1)} - \frac{1/81}{(x-1)^2} - \frac{17/1944}{(x+2)} - \frac{5/324}{(x+2)^2} - \frac{1/54}{(x+2)^3} + \frac{1/1944}{(x-4)}$$

Solve six equations with six unknowns or use a computer algebra system.

When the factors of the denominator in an expression you want to decompose are repeated, you have to modify the partial fraction technique.

Notice here that the first factor is squared and the second is cubed. These factors are repeated.

To decompose the expression, each power of the repeated factor gets its own unknown. Since the second factor was repeated three times, it gets three different fractions in the decomposition.

Solving the partial fraction decomposition is performed in the same way as with non-repeated factors, by finding the common denominator, combining the decomposition into one large expression, and setting terms equal to each other.

Irreducible quadratic factors

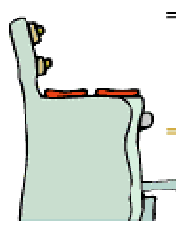
Consider the integral $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\,dx$.

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} = \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2+1)}$$

Notice that $x^2 + 1$ is irreducible. You will have to treat it as one factor.

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+1)}$$

To account for the quadratic factor, put the linear expression $Cx+D$ in the numerator.



$$= \frac{2}{x} + \frac{-1}{x^2} + \frac{3x-2}{(x^2+1)}$$

Moral When you have an irreducible quadratic term in the denominator of a partial fraction, use a linear term of the form $Cx + D$ in the numerator.

Here is another integral to consider. Notice again that the integral cannot be evaluated using u-substitution. So try using partial fractions.

Notice that one of the factors is an irreducible quadratic. These factors require another modification of the partial fractions technique.

When creating unknowns for numerators of the quadratic factors, you must include two unknowns--one unknown multiplied by x and the other a constant.

Solving the decomposition is still performed the same after this step. Combine the fractions and solve for the unknowns.

It is very important that you include the unknown term multiplied by x or you will not get an accurate decomposition.

Unit: Techniques of Integration **Module:** Integration by Partial Fractions with Repeated Factors

Distinct and Repeated Quadratic Factors

Irreducible quadratic factors

Now you have

$$\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx = \int \frac{2}{x}+\frac{-1}{x^2}+\frac{3x-2}{x^2+1}\,dx$$

$$= \int \frac{2}{x}+\frac{-1}{x^2}+\frac{3x}{x^2+1}+\frac{-2}{x^2+1}\,dx$$

Split up the last term.

$$= 2\ln|x| + \frac{1}{x} + \frac{3}{2}\ln\left|x^2+1\right| - 2\arctan x + C$$

Use the integral table.

Since x^2+1 is always positive, you don't need the absolute value symbol in the third term.

$$\boxed{= 2\ln|x| + \frac{1}{x} + \frac{3}{2}\ln\left(x^2+1\right) - 2\arctan x + C}$$

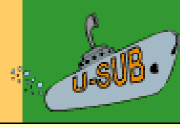
$$\int \frac{-1}{x^2}\,dx = \int -x^{-2}\,dx = \frac{-x^{-1}}{-1}+C = x^{-1}+C = \frac{1}{x}+C$$

$$\int \frac{3x}{x^2+1}\,dx = 3\int \frac{x}{x^2+1}\,dx = \frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln|u|+C = \frac{3}{2}\ln\left|x^2+1\right|+C$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$



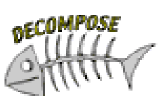
The integral of the decomposition is much easier to evaluate.

Here each integral is evaluated separately. Notice that each piece requires different techniques to solve. It is very important to remember the different integrals you have learned so you can quickly apply them to problems like this one.

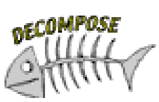
Since the constant of integration is arbitrary, you can combine all of the constants from the individual pieces into one big arbitrary constant. That way you only have to write it once.

Repeated irreducible quadratic factors

Treat repeated quadratic factors just like repeated linear factors.



Decompose $\frac{x+1}{x\left(x^2+3\right)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{\left(x^2+3\right)^2}$



Decompose $\frac{x+1}{x\left(x^2+3\right)^2} = \boxed{\frac{1/9}{x} + \frac{(-1/9)x+0}{x^2+3} + \frac{(-1/3)x+1}{\left(x^2+3\right)^2}}$

If the expression contains irreducible quadratic factors that are raised to a power, then you have to treat those factors just like you would treat a linear factor.

Each power of the irreducible factor gets its own fraction. That means each copy gets two unknowns for quadratic factors.

Partial fraction decompositions can get tricky and involve a lot of algebra. Be very careful when you evaluate them. Watch out for careless errors!

You can always find a common denominator and combine expressions in order to check your algebra.

Unit: Techniques of Integration **Module:** Integration by Partial Fractions with Repeated Factors

Partial Fractions of Transcendental Functions

key concepts:

- If a u-substitution on a **transcendental** integrand leads to a rational expression whose denominator can be factored, use **partial fraction decomposition**.

A peculiar integral

① Consider $\displaystyle\int \frac{e^x}{(e^x-1)(e^x+4)}dx.$

Q How can you crack open this integral?

✘ Integrating directly won't work.

✘ Replacing the denominator by u-substitution doesn't look promising.

A Use u-substitution to replace e^x.

Let $u = e^x$

$du = e^x\,dx$

② $$\int \frac{e^x}{(e^x-1)(e^x+4)}dx = \int \frac{1}{(u-1)(u+4)}du$$

Decompose the integrand into partial fractions.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$= \frac{A(u+4)}{(u-1)(u+4)} + \frac{B(u-1)}{(u-1)(u+4)}$$

$$= \frac{Au + 4A + Bu - B}{(u-1)(u+4)}$$

$$= \frac{(A+B)u + 4A - B}{(u-1)(u+4)}$$ Regroup terms.

Decompose the integrand into partial fractions.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$A + B = 0$ $\quad 4A - B = 1$

$A = -B$

$4(-B) - B = 1$

$-5B = 1$

$B = -\frac{1}{5}$

$A = -\left(-\frac{1}{5}\right)$

$A = \frac{1}{5}$

③ So $\displaystyle\frac{1}{(u-1)(u+4)} = \frac{1/5}{u-1} + \frac{-1/5}{u+4}.$

$$\int \frac{e^x}{(e^x-1)(e^x+4)}dx = \boxed{\frac{1}{5}\ln\left|e^x - 1\right| - \frac{1}{5}\ln\left(e^x+4\right) + C}$$

So far you have only seen **partial fraction decomposition** applied to a rational polynomial function. Now consider this integral.

Traditional methods don't seem to work well here.

But consider this u-substitution.

By making this substitution, you transform a **transcendental** expression into a rational one. Now you can use partial fractions to evaluate the integral.

Start by breaking the integral into its factors with the unknown constants in the numerator.

Find the common denominator.

Combine the expressions.

Regroup the terms.

Now isolate the unknowns and solve.

There are many different ways to solve this system of equations. Here we start by solving for A in terms of B and then substituting.

Once you have the necessary constants, evaluate the simplified integral using other techniques.

So by converting a transcendental expression into a rational expression with a substitution, you can find creative ways to evaluate complicated-looking integrals.