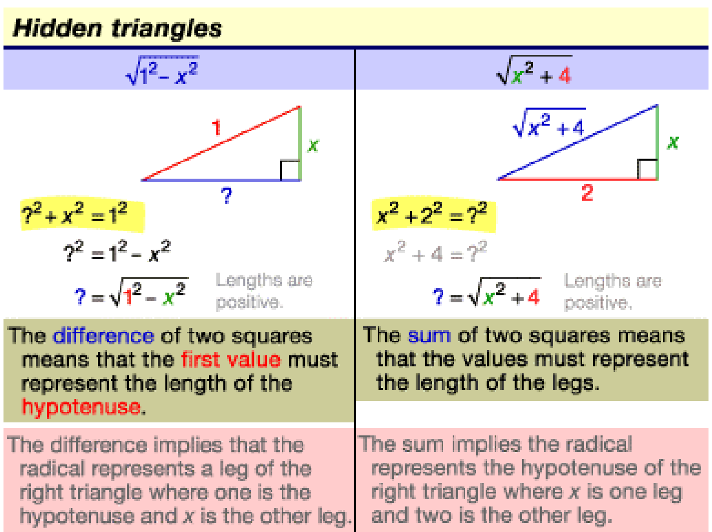
Unit: Techniques of Integration **Module:** Introduction to Trigonometric Substitution

Converting Radicals into Trigonometric Expressions

key concepts:

- Consider the square root of the sum or difference of two squares as information about an unknown right triangle. Trigonometric substitution allows you to convert the square roots into less complicated trigonometric functions.

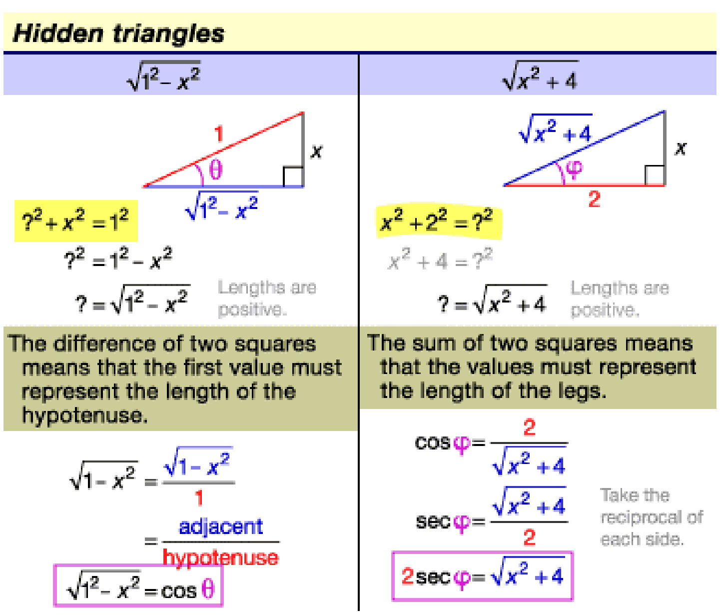


You may encounter radical expressions that contain the sum or difference of squares. You can think of these as the sides of right triangles.

Set the length of one of the sides of the triangle equal to the radical expression. Then you can use the Pythagorean theorem to find expressions for the other two sides.

A radical expression made up of the difference of two squares means the positive term must represent the length of the hypotenuse. The radical expression represents one of the legs.

A radical expression made up of the sum of two squares must represent the hypotenuse of the right triangle. Each of the other two terms represents a leg.



So radical expressions can be thought of as triangles. How does this help you do calculus?

Well, radical expressions can be very complicated to integrate sometimes. A technique to get rid of the radicals would be very helpful.

Now consider the angle in each of the right triangles you have created. Here the angles are marked theta and phi.

But if you have a right triangle and you have angles represented, then you can express the lengths of the sides in terms of trigonometric expressions.

So you could express the first rational expression in terms of cosine and the second in terms of secant.

Now there is no radical in the expression. The new expression might be easier to integrate.

These transformations are called trigonometric substitutions.

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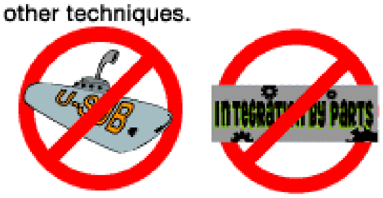
Using Trigonometric Substitution to Integrate Radicals

key concepts:

- Use **trigonometric substitution** to evaluate integrals involving the square root of the sum or difference of two squares.
  1. Match the square root expression with the sides of a right triangle.
  2. Substitute the corresponding trigonometric functions into the integrand.
  3. Evaluate the resulting simpler integral.
  4. Convert from trigonometric functions back to the original variables.
- In a difference of two squares under a radical, the positive term corresponds to the hypotenuse of a right triangle.

Consider $\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$

✘ *U*-substitution will not lead to a simpler integral, nor will integration by parts or other techniques.



✔ The integrand contains the square root of the difference of two squares, which should remind you of a right triangle and trig substitution.

Integrals that include radical expressions can be very tricky to evaluate. Consider this integral.

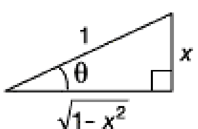
Notice that there is no obvious *u*-substitution to make. Also notice that integration by parts will create successively more complicated integrals.

The presence of a radical does empower you to make **trigonometric substitutions** however.

Trigonometric substitution in action

Consider $\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$

$$\left(\sqrt{1-x^2}\right)^2 + x^2 = 1 - x^2 + x^2 = 1$$



$\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$\frac{d}{dx}\sin\theta = \frac{d}{dx}x$

$\cos\theta\frac{d\theta}{dx} = 1$

$\cos\theta\,d\theta = dx$

$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3\theta}{\cos\theta}\,dx$ Start the trig substitution.

$= \int \frac{\sin^3\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta$ Do not forget to substitute for *dx*.

$= \int \sin^3\theta\,d\theta$ Cancel.

$= \int \sin^2\theta\sin\theta\,d\theta$ Factor out $\sin\theta$.

$= \int \left(1-\cos^2\theta\right)\sin\theta\,d\theta$ Use the Pythagorean formula.

$= \int \left(1-u^2\right)du$ Use *u*-substitution.

$= u - \frac{1}{3}u^3 + C$

$= \frac{1}{3}\cos^3\theta - \cos\theta + C$ Derive the answer or use a table.

Step 4: Converting back to *x*. $= \boxed{\frac{1}{3}\left(\sqrt{1-x^2}\right)^3 - \sqrt{1-x^2} + C}$

To make a trigonometric substitution, start by constructing a triangle with one side that corresponds to the radical in the integrand.

Since the radical is the difference of two perfect squares, then it must correspond to one of the legs of the triangle. Notice that the *x*-term must correspond to the other leg, since it is being subtracted from the other term.

Now that you have constructed your triangle, substitute corresponding trigonometric expressions into the integrand. Notice that sin θ = *x*, according to the triangle you constructed. Cosine can be found the same way. To find *dx*, you must take the derivative of one of your terms. Pick the easiest one.

Once you have used substitution to replace all of the *x*-terms with θs, you can evaluate the integral like a trigonometric expression.

Do not forget the constant of integration. The final step in the evaluation is to convert back to *x*, since that is the original variable.

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Trigonometric Substitutions on Rational Powers

key concepts:

- Use **trigonometric substitution** to evaluate integrals involving the square root of the sum or difference of two squares.
  1. Match the square root expression with the sides of a right triangle.
  2. Substitute the corresponding trigonometric functions into the integrand.
  3. Evaluate the resulting simpler integral.
  4. Convert from trigonometric functions back to the original variables.
- In a sum of two squares under a radical, the radical expression corresponds to the hypotenuse of the right triangle.

Integrating by trigonometric substitution

Consider $\int \frac{dx}{(x^2+1)^{3/2}}$

✗ U-substitution for x^2+1 would require a $2x$ factor, which is not present.

✓ The integrand contains the sum of two squares. The two in the denominator of the exponent implies a square root.

✓ This should remind you of a right triangle and trig substitution.

Consider this integral. Notice that there is no obvious u-substitution that you can make and that other techniques of integration look unpromising.

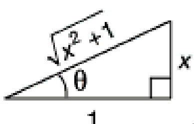
However, the expression in the denominator is raised to a rational power. The two in the denominator of the exponent indicates a square root, just like a radical sign. This integral is a radical expression in camouflaged form.

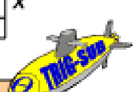
The presence of a radical (or a rational power with a two in the denominator of the exponent) enables you to make a **trigonometric substitution**.

Integrating by trigonometric substitution

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{1}{\left(\sqrt{x^2+1}\right)^3}\,dx$$

Step 1 Finding a matching triangle





$$\cos\theta = \frac{1}{\sqrt{x^2+1}}$$

$$\tan\theta = x$$

$$\frac{d}{dx}(\tan\theta) = \frac{d}{dx}(x)$$

$$\sec^2\theta\frac{d\theta}{dx} = 1$$

$$\sec^2\theta\, d\theta = dx$$

Step 2 Substituting the trig functions

$$= \int \cos^3\theta\, dx$$

$$= \int \cos^3\theta \sec^2\theta\, d\theta$$

$$= \int \cos^3\theta \cdot \frac{1}{\cos^2\theta}\, d\theta$$

Secant is the reciprocal of cosine.

$$= \int \cos\theta\, d\theta$$

Step 3 Evaluating the integral

$$\int \cos\theta\, d\theta = \sin\theta + C$$

Step 4 Converting back to x

$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

$$\text{So } \int \frac{dx}{(x^2+1)^{3/2}} = \frac{x}{\sqrt{x^2+1}} + C.$$

To make a trigonometric substitution, start by constructing a triangle with one side that corresponds to the expression raised to the rational power in the integrand.

Since this expression is the sum of two perfect squares, the radical must correspond to the hypotenuse of a right triangle. Each individual term therefore represents one of the legs.

Now that you have constructed your triangle, substitute corresponding trigonometric expressions into the integrand. Notice that to find dx, you probably want to differentiate the tangent expression, since it will be easier than the cosine expression.

Now that the integral is expressed in terms of θ instead of x, it is much easier to integrate.

The last step is to write the answer in terms of x.