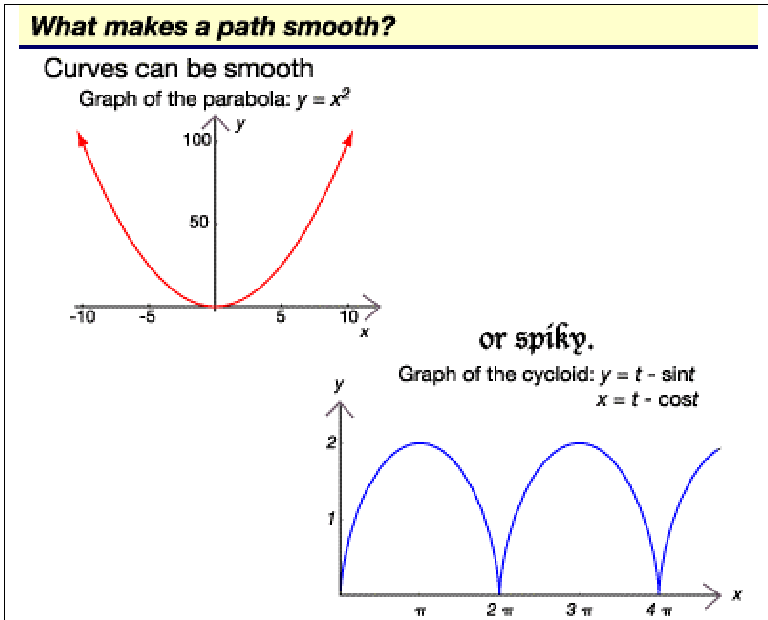
Derivatives of Parametric Equations

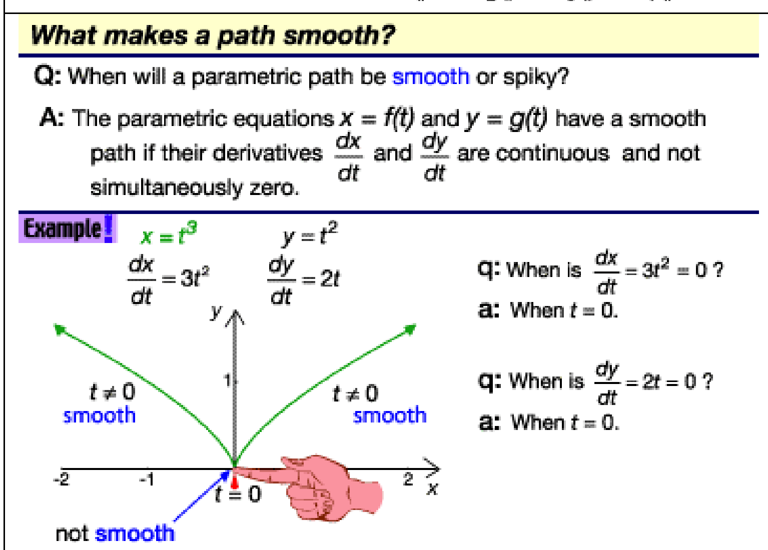
key concepts:

- If f and g are continuous functions of t, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations and t is called the parameter.
- A curve represented by $x = f(t)$ and $y = g(t)$ on an interval I is **smooth** if f' and g' are continuous on I and not simultaneously zero, except perhaps at the endpoints of I.
- The formula $\frac{dy}{dx} = \frac{dy}{dx} \Big/ \frac{dx}{dt}$ where $\frac{dx}{dt} \neq 0$ is a convenient way to find the derivative of a parameterized curve.



Curves can be **smooth** or spiky. A curve is smooth if it has no sharp turns or points.

So how do you tell if a parametric curve is spiky?



Curves can be spiky where their first derivative is undefined. So you could convert the parametric equations into rectangular form and take the derivative to see if the curve spikes. But converting to rectangular form is time consuming.

Instead you can just take the derivative of the parametric equations with respect to the parameter.

The curve will be smooth everywhere that the two derivatives are continuous and not simultaneously zero. If only one of the derivatives equals zero the curve is still smooth.

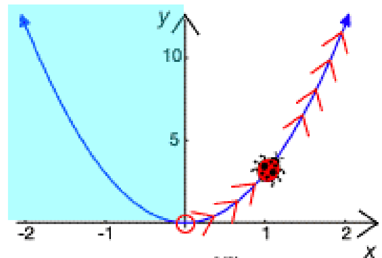
The reason the curve can get spiky here is because both derivatives are zero. This gives the curve a chance to change directions.

Derivatives of Parametric Equations

The direction of the bug

Q: Which way will the bug crawl?

Example! $x = e^t \quad y = 3e^{2t}$



A: At $t = 0$, $x = e^0 = 1$ and $y = 3e^{2(0)} = 3(1) = 3$, so the bug is located at (1,3).

As time moves forward, or t increases, the bug moves to higher values of x and y or to the right on the path.

As time moves backwards, or t becomes more negative, the bug approaches, but never reaches, the point (0,0).

Note that $\lim_{t\to-\infty} e^t = 0$ and $\lim_{t\to-\infty} 3e^{2t} = 0$.

The bug never reaches the part of the curve to the left of the origin.

Another important aspect of parametric curves is how the parameter can affect the domain and range.

Notice in this example that eliminating the parameter produces a parabola.

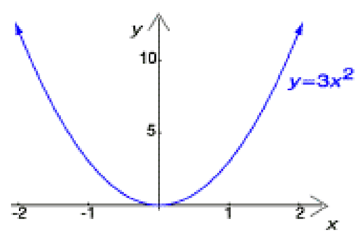
However, because the exponential function is always positive, there is no way for the x-term to ever be negative.

As such, you cannot plot a point to the left of the origin using the parametric equations.

Parametric equations plot a position on a path. They do not have to use the entire path to do so.

The derivative of parametric equations

Example! $x = e^t \quad y = 3e^{2t}$



Q: What is $\frac{dy}{dx}$?

A: *Method 1* $\frac{dy}{dx} = 6x$

A: *Method 2*

Using the chain rule

$$\frac{dy}{dt} = \frac{d}{dt}[3e^{2t}] = 3\frac{d}{dt}[e^{2t}] = 3e^{2t}\frac{d}{dt}[2t] = (2)3e^{2t} = 6e^{2t}$$

$$\frac{dx}{dt} = \frac{d}{dt}[e^t] = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = 6e^{2t}/e^t = 6e^t = 6x$$

Recall that $x = e^t$

The formula $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$ (as long as $\frac{dx}{dt} \neq 0$) is a convenient way to find the derivative of parametric curves.

Suppose you wanted to find the first derivative of these parametric equations.

You could express the parametric equations in rectangular form and take the derivative of that equation. But that takes a lot of algebra.

Instead, you can use the parametric equations to find the derivative through an application of the chain rule.

According to the chain rule, $\frac{dy}{dx} = \frac{dy}{dx}\Big/\frac{dx}{dt}$.

So if you take the derivative of each parametric equation and divide the derivative of y by the derivative of x, you get the derivative of y with respect to x.

Graphing the Elliptic Curve

key concepts:

- The formula $\frac{dy}{dx} = \frac{dy}{dx} \Big/ \frac{dx}{dt}$ where $\frac{dx}{dt} \neq 0$ is a convenient way to find the derivative of a parameterized curve.

Parametric representation of an elliptic curve	
Example! $x = t^2$ $y = t^3 - 3t$	How can you graph parametric equations?
Three methods of graphing curves:	One option is to plot points. But plotting points doesn't give you very accurate curves.
Technique 1: Plug in values of t and calculate x- and y-values of points in the curve. It'll work, but it is a slow process.	
Technique 2: Eliminate the parameter and derive an equation in the familiar form $y = f(x)$. Can become complicated and time consuming.	Another option is to eliminate the parameter and put the equations in rectangular form. But eliminating the parameter is not always easy to do.
Technique 3: Use the power of calculus. The best choice for curve sketching, of course!	The best option for graphing parametric equations is to use curve sketching techniques from calculus. By modifying the techniques you learned in the first semester of Calculus you can draw very accurate curves for parametric equations.
Use the formula $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ as long as $\frac{dy}{dx} \neq 0$ to find the derivative of the curve.	
$\frac{dy}{dt} = 3t^2 - 3$ $\qquad \frac{dx}{dt} = 2t$ $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$	To sketch the curve, start by finding the derivatives.
The maxima and minima of an elliptic curve	
Example! $x = t^2$ $y = t^3 - 3t$	
The maxima and minima of the curve occur when the derivative is zero.	Critical points still occur when the derivative of y with respect to x is undefined or equal to zero.
Q: When is $\frac{dy}{dx} = 0$?	Set the derivative equal to zero and solve.
A: When the numerator of $\frac{3t^2 - 3}{2t}$ is zero.	
$3t^2 - 3 = 0$ $3(t^2 - 1) = 0$ Factor out the 3. $3(t+1)(t-1) = 0$ Factor the difference of perfect squares. $(t+1)(t-1) = 0$ Divide by 3.	Notice that the derivative is in terms of the parameter, t. This is useful, since the graph might not be a function of x but the parametric equations are both functions of t. Each value of the parameter corresponds to a unique point.
$t = -1$, $t = 1$ ← times when there may be a max or a min	
Q: What points of the curve correspond to $t = 1$ and $t = -1$? **A:** Plug $t = -1$ and $t = 1$ into the parametric equations.	Now you have the times where the graph could hit a maximum or minimum point. Plug this value into the parametric equations to determine the actual points.
$x(-1) = (-1)^2 = 1$ $\qquad y(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$	So the points $(1, 2)$ and $(1, -2)$ are potential extreme values.
$x(1) = 1^2 = 1$ $\qquad y(1) = 1^3 - 3(1) = 1 - 3 = -2$	

Graphing the Elliptic Curve

Undefined slopes of tangents to elliptic curves

Example! $x = t^2 \quad y = t^3 - 3t$

The slope of the tangent to the elliptic curve is undefined when the denominator of the derivative is zero.

Q: When is the denominator of $\frac{dy}{dx}$ equal to zero?

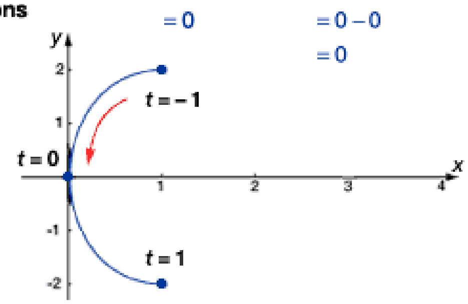
A: When $2t$ is 0. $\quad 2t = 0$
$t = 0$

Q: Where is the curve located at $t = 0$?

A: Plug $t = 0$ into the parametric equations

$$x(0) = 0^2 = 0 \qquad y(0) = 0^3 - 3(0) = 0 - 0 = 0$$

The origin is part of the curve, and at that location the curve is tangent to the y-axis.



Remember that critical points also occur when the derivative is undefined. The derivative is undefined when its denominator is equal to zero.

This occurs when $t = 0$.

Plugging this critical point into the parametric equations gives you the point $(0,0)$.

So $t = 1$ corresponds to the point $(1,2)$. $t = 0$ corresponds to the point $(0,0)$. $t = 1$ corresponds to the point $(1,-2)$.

Locations where the elliptic curve crosses the x-axis

Example! $x = t^2 \quad y = t^3 - 3t$

Q: Where does the elliptic curve cross the x-axis?

A: At $y = 0$.

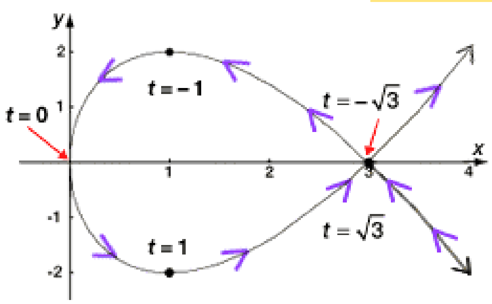
$$t^3 - 3t = 0$$
$$t(t^2 - 3) = 0$$
$$t(t - \sqrt{3})(t + \sqrt{3}) = 0$$
$$t = 0, t = \pm\sqrt{3}$$

Q: What points of the curve correspond to $t = \pm\sqrt{3}$?

A: Plug $t = \pm\sqrt{3}$ into the x-equation.

$$x(\pm\sqrt{3}) = \pm\sqrt{3}^2 = 3$$

The curve crosses the point (3,0) twice, when $t = \sqrt{3}$ and when $t = -\sqrt{3}$.



Q: What is the direction of the elliptic curve?

A: At $t = 0$, the curve is at the origin.
As t increases, the curve moves in the downward direction.
As t decreases or becomes more negative, the curve moves in the upward direction.

The graph of the elliptic curve includes the path direction.

To find when the graph crosses the x-axis, find where the parametric equation for y is equal to zero.

The parametric equation crosses the axis when t equals the negative square root of three, zero, and the square root of three.

Plugging those values into the parametric equation for x completes the point.

Now you have a pretty good idea of the behavior of the graph. There is a maximum point at $t = -1$, a minimum point at $t = 1$, a vertical tangent at $t = 0$, and the graph crosses the axis twice at $x = 3$.

You can even determine the direction of the path by considering where the points are for different values of the parameter.

Unit: Parametric Equations and Polar Coordinates **Module:** Calculus and Parametric Equations

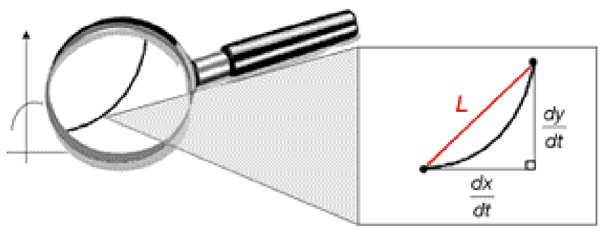
Arc Length of a Parameterized Curve

key concepts:

- **Arc length** is the length of a curve.
- Given a smooth curve described by the parametric equations $x = f(t)$ and $y = g(t)$, the arc length L between the points a and b is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Looking through the magnifying glass



Consider the straight line because the curve is too complicated.

Q: What is the length L of the line?

A: Make the line the hypotenuse of a right triangle and use the Pythagorean theorem.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = L^2$$

$$L = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

As L becomes smaller and smaller, it approximates the curve more accurately. Assume that L becomes infinitely small.

One of the applications of calculus is finding **arc length**. Arc length is the length of a curve.

Arc length is a little easier to understand when dealing with parametric equations instead of the rectangular equation. If you look at a small section of the curve you will see that the curve can be approximated by a line. Find the length of the line with the Pythagorean theorem.

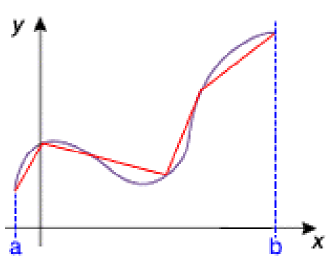
Since the change in the x-direction is just a small change in x and the change in the y-direction is a small change in y, then the lengths of the legs of the triangle can be expressed as the derivatives of the respective parametric equations.

As you make the distance between the points smaller and smaller, L becomes more and more accurate.

The formula for arc length

To find the length of a curve, you need to sum up the lengths of all the secant lines of the curve. Since there are an infinite number of terms in the sum, compute the length as an integral.

$$\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



The formula directly follows from this relationship. By summing up all these infinitely small pieces you find the length of the curve.

Very Important

The formula for arc length does not represent the area under the curve.

When computing the arc length from parametric equations, integration is with respect to the parameter (such as t).

Remember, the arc length does not represent the area under a parametric curve. It represents the length of the curve.

Calculating the arc length by integrating with respect to the parameter can be much easier than using the rectangular form.

Finding Arc Length

key concepts:

- Given a smooth curve described by the parametric equations $x = f(t)$ and $y = g(t)$, the arc length L between the points a and b is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Review of arc length

For a parameterized curve,

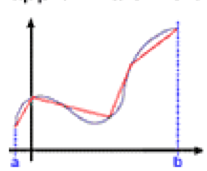
$$\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

alternatively written as:

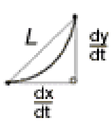
$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

To find the arc length of a parameterized curve, start by taking the derivatives of the parametric equations.

Lines of length L approximate the curve:



Each line is the hypotenuse of a triangle with length L :



so that: $L = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

You don't need the derivative of y with respect to x because it is the change with respect to time that determines the lengths of the legs of the arbitrary triangle you will use to calculate the arc length.

When the lengths L become infinitely small, the sum of the lengths approaches the **arc length**.

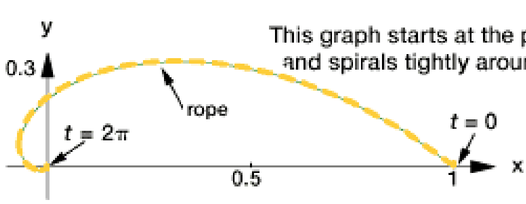
By integrating, you sum up all of the infinitely small lengths to produce one value.

Estimating arc length

Example! $x(t) = e^{-t}\cos t \quad y(t) = e^{-t}\sin t$

Q: What is the graph of these parametric equations?

A:



This graph starts at the point (1,0) and spirals tightly around the origin.

It is a good idea to graph the parametric equations before you find the arc length.

In general, this graph doesn't have to be very precise. However, if you have a good graph you can approximate the length you are going to calculate.

At $t = 0$, $\quad x(0) = e^{-0}\cos 0 = 1(1) = 1 \qquad y(0) = e^{-0}\sin 0 = 1(0) = 0$

Q: What is the length of the arc that is swept out by the parametric equations as t goes from 0 to 2π?

A: Use a piece of rope to trace out the curve. Measure the rope.

arc length = 1.4 ← estimate

Guess = 1.4

Estimate the arc length by stretching a string or rope along the curve. When you straighten the rope, you can measure the straight line.

Finding Arc Length

Calculating arc length

Example! $x(t)=e^{-t}\cos t \quad y(t)=e^{-t}\sin t$

$$=\int_0^{2\pi}\sqrt{\left[-e^{-t}(\sin t+\cos t)\right]^2+\left[e^{-t}(\cos t-\sin t)\right]^2}\,dt$$

Once you have the derivatives you are ready to plug them into the formula and begin integrating.

$$=\int_0^{2\pi}\sqrt{e^{-2t}(\sin^2 t+2\sin t\cos t+\cos^2 t)+\left[e^{-t}(\cos t-\sin t)\right]^2}\,dt$$

Here the first expression has been expanded.

$\sin^2 t+\cos^2 t=1$

$$=\int_0^{2\pi}\sqrt{e^{-2t}(2\sin t\cos t+1)+\left[e^{-t}(\cos t-\sin t)\right]^2}\,dt$$

Using the Pythagorean identity you can combine the sine-squared and the cosine-squared term.

Square and FOIL.

$$=\int_0^{2\pi}\sqrt{e^{-2t}(2\sin t\cos t+1)+e^{-2t}(\cos^2 t-2\sin t\cos t+\sin^2 t)}\,dt$$

Now expand the second expression.

$\sin^2 t+\cos^2 t=1$

$$=\int_0^{2\pi}\sqrt{e^{-2t}(2\cancel{\sin t\cos t}+1)+e^{-2t}(-2\cancel{\sin t\cos t}+1)}\,dt$$

Use the Pythagorean identity again. Notice that you can cancel the trigonometric factors by distributing.

Distribute.

$$=\int_0^{2\pi}\sqrt{2e^{-2t}}\,dt$$

$$=\int_0^{2\pi}\sqrt{2e^{-2t}}\,dt$$

Combine terms.

Separate the terms underneath the radical and move the constant outside the integral.

$$=\sqrt{2}\int_0^{2\pi}e^{-2t/2}\,dt$$

Move constant out of integral.

Simplify the exponential function.

$$=\sqrt{2}\int_0^{2\pi}e^{-t}\,dt$$

A quick *u*-substitution gives you the antiderivative.

Reduce fraction in exponent.

$$=-\sqrt{2}e^{-t}\Big|_0^{2\pi}$$

Integrate.

$$=-\sqrt{2}e^{-2\pi}-(-\sqrt{2})$$

Now plug in the limits of integration.

Evaluate.

$$=\sqrt{2}-\sqrt{2}e^{-2\pi}$$

$$=\sqrt{2}\left(1-\frac{1}{e^{2\pi}}\right)$$

The guess of 1.4 was pretty close to the actual value!

$$=1.41421\left(1-\frac{1}{535.492}\right)=1.41421(0.998133)=1.41157$$

Guess = 1.4

Using a calculator gives a decimal value for the arc length. Notice that the initial guess was pretty close. That's a good clue that you didn't make any algebra mistakes.