Monotonic and Bounded Sequences

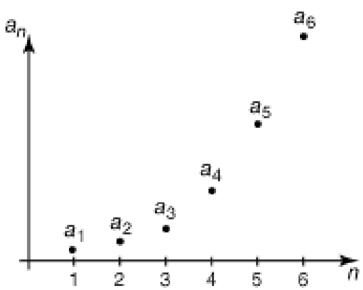
key concepts:

- A sequence is **increasing** if each term is larger than the previous one. A sequence is **decreasing** if each term is smaller than the previous one.
- A sequence is **monotonic** if the terms are **nonincreasing** ($a_1 \le a_2 \le \cdots \le a_n \le \cdots$) or **nondecreasing** ($a_1 \ge a_2 \ge \cdots \ge a_n \ge \cdots$). Increasing or decreasing sequences are **strictly monotonic**.
- A sequence is **bounded from above** if there is a value that the terms never exceed. A sequence is **bounded from below** if there is a value below which the terms never fall. A **bounded** sequence is bounded from above and below.
- If a sequence is monotonic and bounded, then it has a limit.

Monotonicity

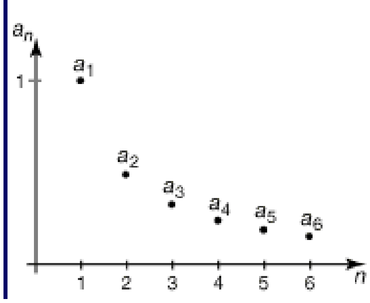
The sequence $\{n^2\}$ is increasing.

$\{n^2\} = \{1, 4, 9, 16, \ldots\}$



The sequence $\left\{\frac{1}{n}\right\}$ is decreasing.

$\left\{\frac{1}{n}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$



If each successive term of a sequence is larger than the previous term, then the sequence is **increasing**.

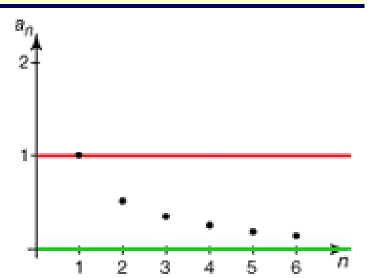
If each successive term of a sequence is smaller than the previous term, then the sequence is **decreasing**.

Some sequences aren't quite increasing or decreasing, but instead never decrease or never increase. These sequences are called **monotonic** sequences. A monotonic sequence can repeat terms, but can never change directions.

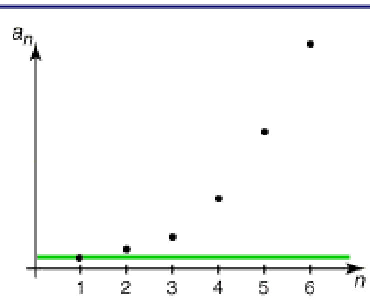
A monotonic sequence that doesn't repeat numbers is called **strictly monotonic**.

Bounded sequences

In fact, the sequence $\left\{\frac{1}{n}\right\}$ is bounded above and bounded below.



However, the sequence $\{n^2\} = \{1, 4, 9, 16, \ldots\}$ is not bounded above, but it is bounded below, by one.



Another way you can analyze or describe a sequence is to find numbers that the terms of the sequence are always less than or greater than.

If the terms of a sequence are always less than or equal to a particular number, then that sequence is **bounded above** by that number.

If the terms of a sequence are always greater than or equal to a particular number, then that sequence is **bounded below** by that number.

A sequence that is bounded above and bounded below is called a **bounded** sequence.

Sequences do not have to be bounded. In fact, it is possible to create a sequence that is neither bounded above or below.

Monotonic and Bounded Sequences

Bounded sequences

Example!

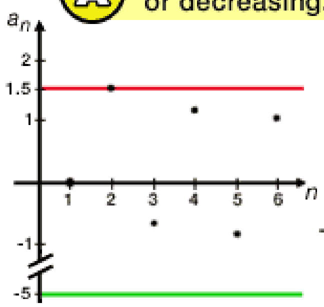
$$\left\{(-1)^n + \frac{1}{n}\right\} = \left\{-1+\frac{1}{1}, 1+\frac{1}{2}, -1+\frac{1}{3}, \ldots\right\}$$

oscillates between -1 and 1

$$= \left\{0, \frac{3}{2}, -\frac{2}{3}, \ldots\right\}$$

Q Is this sequence increasing or decreasing?

A The sequence is neither increasing or decreasing. It is not monotonic.



This sequence is bounded above by one and a half. Its terms will never exceed that value.

The sequence is also bounded below by –5, because the terms will never be smaller than that value.

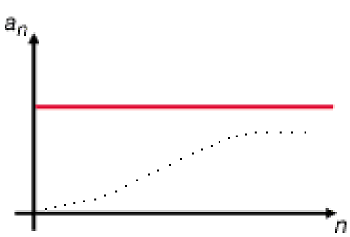
Consider this sequence. The terms are defined by a sum. Writing out some of the terms of the sequence will give you a better idea of how the sequence behaves.

Notice that the $(-1)^n$ makes the sequence jump around. Looking at the first three terms proves that the sequence is not increasing. Looking at the first two terms proves that the sequence is not decreasing. The sequence isn't monotonic either, since it isn't nonincreasing or nondecreasing. Every other pair of terms switches from increasing to decreasing.

The sequence is bounded above and below however. Since the sequence is bounded above and below, it is a bounded sequence.

A bounded monotonic sequence

one last note!



The sequence is increasing, but it also tapers off. Therefore its terms cannot approach infinity and they cannot bounce around several values. The terms must approach a value.

The limit of the sequence exists.

A sequence that is decreasing and bounded from below also has a limit.

It might seem like being monotonic or bounded really doesn't tell you much about a sequence. But as it turns out, these properties are very important.

Suppose a sequence was increasing and bounded. Since the sequence is bounded, there must be a number that the sequence never reaches. But since the sequence is increasing, it can never turn back. So as you look at higher-indexed terms, the sequence has to approach a value. Therefore, bounded increasing sequences must have a limit.

A similar argument can be made for decreasing sequences.

In fact, any bounded monotonic sequence will have a limit.