Unit: Sequences and Series **Module:** Polynomial Approximations of Elementary Functions

Polynomial Approximation of Elementary Functions

key concepts:

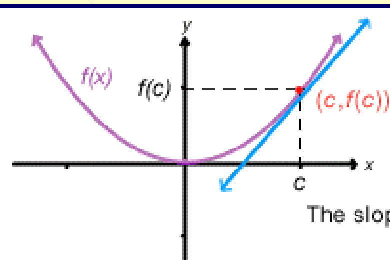
- Polynomials can approximate complicated functions. The **tangent line approximation** of $f(x)$ at $x = c$ is $y = f(c) + f'(c)(x - c)$.
- Near $x = c$, the tangent line is a good approximation to the curve of $f(x)$. However, for values further away from c the approximation is not so good.

Evaluating complicated functions

Q: Can you evaluate sin 4 without a calculator?

Although it seems very simple, sine is a difficult function to evaluate. Sure, you can evaluate the sine of special angles, but what is the sine of a number like four?

The tangent line approximation



The slope of the tangent at $x = c$ is $f'(c)$.

Q: What is an approximation of this curve at $x = c$?

A: The tangent to the curve at $x = c$ approximates the curve very well.

The point-slope form of a line is $y - y_1 = m(x - x_1)$

$$y = y_1 + m(x - x_1)$$

The tangent line approximation of $f(x)$ at $x = c$: $y = f(c) + f'(c)(x - c)$

Sine may be hard to evaluate, but polynomials are not. Polynomials can be evaluated with just addition and multiplication. And one of the simplest polynomials is the expression of a line.

A simple, yet accurate method for approximating a function is the **tangent line approximation**.

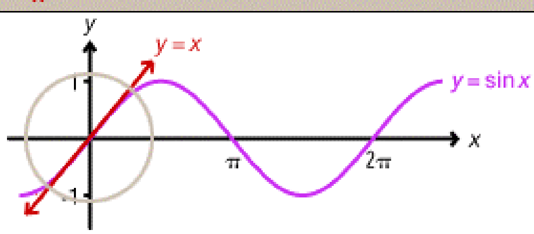
You can use the point-slope form of a line to determine the formula for the tangent line.

To find the tangent line approximation of a function f at a point c, you will need to know the slope of the line tangent to the curve at c and the value of f at c.

Example: Approximating sin x

The tangent line at $x = 0$ is

$$\begin{aligned} y &= f(0) + f'(0)(x - 0) \\ &= \sin 0 + (\cos 0)x \\ &= 0 + 1x \\ &= x \end{aligned}$$



Since you know the value of sine of zero, you can find the tangent line approximation of sine at zero. Just plug in the values of sine, its derivative, and the point zero.

Now you can approximate other values of sine near zero.

Inside this circle you can see that the line matches up well with the graph of sine. The further away you go from the origin, the worse the approximation becomes. But there are other polynomials besides linear ones. Maybe a quadratic polynomial, which describes a parabola, will fit the curve of sine better.

Unit: Sequences and Series **Module:** Polynomial Approximations of Elementary Functions

Higher Degree Approximations

key concepts:

- The quadratic approximation of $f(x)$ at the point $x = c$ is
  $$f^{(0)}(c)(x-c)^0 + f^{(1)}(c)(x-c)^1 + \frac{f^{(2)}(c)(x-c)^2}{2}.$$
- Higher-degree **polynomial approximations** result in more accurate representations.

linear approximation of $f(x)$ at c: $y = f^{(0)}(c)(x-c)^0 + f'(c)^{(1)}(x-c)^1$

Q: What is the quadratic approximation of $f(x)$ at c?

A: It must pass through the point $(c, f(c))$ and $y'(c) = f'(c)$. Start with the linear approximation and add terms to make sure the second derivatives agree.

$$y = f^{(0)}(c)(x-c)^0 + f^{(1)}(c)(x-c)^1 + \frac{f^{(2)}(c)(x-c)^2}{2}$$

You can rewrite the tangent line approximation $y = f(c) + f'(c)(x-c)$ so that it shows a pattern. Note that $f^{(0)}$ is f and anything raised to the zero power equals one.

Extending this pattern leads to the quadratic approximation. But remember that the derivatives must agree. When you differentiate, the squared term will produce a factor of two. You have to compensate with a two in the denominator.

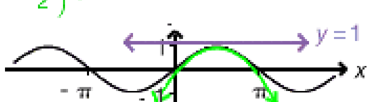
Example: Consider $y = \sin x$

$$y\left(\frac{\pi}{2}\right) = f^{(0)}\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)^0 + f^{(1)}\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)^1 + \frac{f^{(2)}(\pi/2)}{2}\left(x-\frac{\pi}{2}\right)^2$$

$$y\left(\frac{\pi}{2}\right) = \left(\sin\frac{\pi}{2}\right)(1) + \left(\cos\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right) + \frac{-(\sin \pi/2)}{2}\left(x-\frac{\pi}{2}\right)^2$$

$$y\left(\frac{\pi}{2}\right) = (1)(1) + (0)\left(x-\frac{\pi}{2}\right) + \frac{(-1)(x-\pi/2)^2}{2}$$

$$y\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 \longleftarrow \text{quadratic approximation}$$



$$y = 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2$$

The linear approximation of sine at $\pi/2$ produced a horizontal line described by $y = 1$. Use the formula for the quadratic approximation to try for a better fit.

When you transform this expression into the standard form, you will see that the squared term has a negative sign in front. That means you have a sad-faced parabola, as shown.

Notice that the quadratic hugs the sine curve much better than the line does.

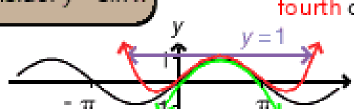
Q: What is the fourth power approximation?

A: $$y = f^{(0)}(c)(x-c)^0 + \frac{f^{(1)}(c)}{1!}(x-c)^1 + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

If you keep following the pattern, you can derive formulas for higher-order approximations. Notice that if you take the derivative of the cubed term you will introduce a factor of three. When you differentiate again, you will introduce a factor of two. To compensate you need to divide by three times two, which is three factorial (3!).

Example: Consider $y = \sin x$

$y = \sin x$ and its first, second, and fourth order polynomial approximations



$y = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{24}(x - \frac{\pi}{2})^4$ $\quad y = 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2$

The cubic approximation is actually the same as the quadratic, because the cubed term has a coefficient of zero.

The fourth order **polynomial approximation** forms a "w" as it hugs the sine curve even better than the quadratic.