Calculus Lecture Notes

thinkwell ©2001

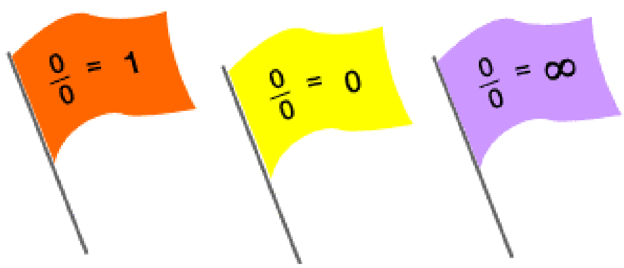
Unit: L'Hôpital's Rule **Module:** Indeterminate Quotients

Indeterminate Forms

key concepts:

- A limit of a function is called an **indeterminate form** when it produces a mathematically meaningless expression. Indeterminate forms are also called **indeterminant forms**. Two types of indeterminate forms are $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
- Some indeterminate forms can be solved by using algebra tricks such as canceling or dividing by the highest power of *x*.

$\frac{0}{0} = ?$



When taking limits, sometimes you will encounter expressions whose meanings can be interpreted in different ways. These limits are called **indeterminate forms**. 0/0 is one example.

One camp says that the indeterminate form equals one because it is a number divided by itself.

Another says that zero divided by anything is zero.

A third says that any number divided by zero is infinity.

Similar arguments hold for the form ∞/∞.

Example! Evaluate $\lim_{x\to 3} \frac{2x^2-18}{x-3}$

$$\lim_{x\to 3} \frac{2x^2-18}{x-3} \to \frac{2(3)^2-18}{(3)-3} \to \frac{0}{0}$$

Indeterminate form! Need to do more work.

$$\frac{2x^2-18}{x-3} = \frac{2(x^2-9)}{x-3}$$

Factor.

$$= \frac{2(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

Factor the difference of two perfect squares.

$$= 2(x+3),\ x \neq 3$$

$$\lim_{x\to 3} \frac{2x^2-18}{x-3} = \lim_{x\to 3} 2(x+3)$$

$$\boxed{=12}$$

Remember: a limit means only *approaching* three

When an indeterminate form arises, you will have to do more work.

One algebraic trick involves factoring the numerator and the denominator.

In this case, you can cancel the (*x* – 3) factors as long as you promise not to let *x* be equal to three.

Example!

$$\lim_{x\to\infty} \frac{-x^3+9x-5}{10x^2+3} = \lim_{x\to\infty} \frac{-1+(9/x^2)-(5/x^3)}{(10/x)+(3/x^3)}$$

$$= \frac{-1+(9/\infty)-(5/\infty)}{(10/\infty)+(3/\infty)}$$

$$= \frac{-1+0-0}{0+0} = \boxed{-\frac{1}{0}} = -\infty$$

NOT an indeterminate form

To evaluate this limit, look for the highest power.

In this case, x^3 is the highest power, so divide the numerator and denominator by it. You are essentially multiplying by a form of one.

Now all the terms have *x* in the denominator, except one. Those terms will approach zero.

The result is not an indeterminate form. It is –∞.

Indeterminate Forms

An Intro to L'Hôpital's Rule

key concepts:

- A limit of a function is called an **indeterminate form** when it produces a mathematically meaningless expression. Indeterminate forms are also called **indeterminant forms**.
- **L'Hôpital's rule** enables you to evaluate indeterminate forms quickly by using derivatives.

Indeterminate forms

The indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ require you to resort to algebra tricks to solve for their respective limits.

$$\lim_{x\to 3}\frac{2x^2-18}{x-3}\ \boxed{=12}$$ Factor.

$$\lim_{x\to\infty}\frac{-x^3+9x-5}{10x^2+3}\ \boxed{=-\infty}$$ Isolate highest power and solve.

For simple functions, their limits can be evaluated easily with simple algebra and direct numerical substitution.

For complicated functions it might not be easy to factor or simplify their respective limits, and in some cases algebraic manipulation can be time consuming and lead to errors.

STEPS FOR USING L'HÔPITAL'S RULE:

1. Make sure you have an indeterminate form.
   $\frac{0}{0}$ or $\frac{\infty}{\infty}$
2. Take the derivatives of the numerator and the denominator.
3. Take the limit of this new quotient.

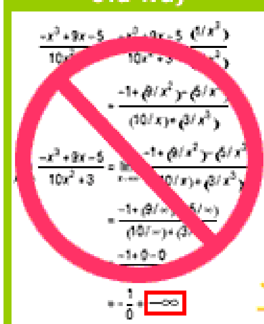
Suppose a function has a numerator and a denominator and its limit produces an **indeterminate form**. Then you can apply **L'Hôpital's rule**.

If the top and the bottom are differentiable, the limit of the function is equal to the limit of the derivative of the numerator divided by the derivative of the denominator.

Example!

Evaluate $\lim_{x\to\infty}\frac{-x^3+9x-5}{10x^2+3}$

Old way



New way (use L'Hôpital's rule)

$$\lim_{x\to\infty}\frac{-x^3+9x-5}{10x^2+3}\to\frac{-\infty}{\infty}$$ indeterminate form!

$$=\lim_{x\to\infty}\frac{-3x^2+9}{20x}\to\frac{-\infty}{\infty}$$ indeterminate form!

$$=\lim_{x\to\infty}\frac{-6x}{20}\ \boxed{=-\infty}$$

Use the rule again. Evaluate.

Use the rule only with indeterminate forms.

Instead of using algebra, L'Hôpital's rule enables you to evaluate indeterminate forms quickly by using derivatives.

If the quotient of the derivatives still yields an indeterminate form, L'Hôpital's rule can be applied again.

To avoid mistakes, make sure the limit produces an indeterminate form before using L'Hôpital's rule.

CAUTION

Q: Should you use the quotient rule when taking the derivative?

A: L'Hôpital's rule doesn't instruct you to take the derivative of the function. The rule says to take the derivative of the numerator and divide that by the derivative of the denominator.

CAUTION: Notice that in the example above, the quotient rule is not applied when taking the derivatives. L'Hôpital's rule doesn't instruct you to take the derivative of the function. This makes L'Hôpital's rule easier to use since you do not have to remember the quotient rule.

Unit: L'Hôpital's Rule **Module:** Indeterminate Quotients

Basic Uses of L'Hôpital's Rule

key concepts:

- When evaluating a limit, it is always a good idea to plug in the value first. If the result yields an **indeterminate form**, then use **L'Hôpital's rule**.
- As long as the limit is still an indeterminate form you can reuse L'Hôpital's rule.
- L'Hôpital's rule might not produce the right answer if you use it on a limit that does not produce an indeterminate form.

A basic example

Example!

Evaluate $\lim_{x \to -2} \frac{x^2+3x+2}{x^2+x-2}$

$$\lim_{x \to -2} \frac{x^2+3x+2}{x^2+x-2} \to \frac{(-2)^2+3(-2)+2}{(-2)^2+(-2)-2}$$ Start by plugging in.

$$\to \frac{0}{0}$$ indeterminate form!

L'Hôpital's rule $\frac{\text{derivative of top}}{\text{derivative of bottom}}$ & take the limit again

Make sure to plug the value –2 in place of x as your first step. Remember, you can only use **L'Hôpital's rule** if the limit produces the **indeterminate form** $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

Once you have verified that the limit produces an indeterminate form, L'Hôpital's rule works by taking the derivative of the numerator and the derivative of the denominator. These form a new limit that may be easier to evaluate.

A mean example

Example! take a guess

Evaluate $\lim_{x \to \infty} \frac{x^{103}-1000}{103x+2x^{103}}$

$$\lim_{x \to \infty} \frac{x^{103}-\cancel{1000}}{\cancel{103x}+2x^{103}} \; ? \; \frac{1}{2}$$

As the numerator and denominator race towards infinity, the -1000 and the 103x are not significant.

Before using L'Hôpital's rule, make a guess as to the value of the limit. It's okay if your guess is wrong—guessing will help you learn the patterns that lead to various answers.

This is a mean example of a limit because the higher-power terms are not in the front. Terms with higher powers have a greater effect on the limit when x approaches infinity.

Since –1000 and 103x have relatively low powers, their effect is negligible. The remaining terms are both raised to the 103rd power, so they will cancel each other out and leave only their coefficients. For this reason 1/2 is a good guess.

A role-playing exercise

Example!

Evaluate $\lim_{x \to \infty} \frac{15}{x^2+1}$

For this example, Professor Burger played the role of a student who applied L'Hôpital's rule.

However, if you plug in first, you will see that this limit does not produce an indeterminate form. Evaluating the limit directly will be less work and will give an answer you can be sure of. Using L'Hôpital's rule incorrectly might result in a wrong answer.

Unit: L'Hôpital's Rule **Module:** Indeterminate Quotients

More Exotic Examples of Indeterminate Forms

key concepts:

- As long as the limit still produces an **indeterminate form** you can reuse **L'Hôpital's rule**.
- When applying L'Hôpital's rule to a quotient containing one or more products or compositions of functions, it is necessary to use the product or chain rules.
- L'Hôpital's rule might not give you the right answer if you use it on a limit that does not produce an indeterminate form.

Classic limits

Example!

$$\lim_{x\to 0}\frac{\sin x}{x} \to \frac{0}{0}$$

Start by plugging in. Indeterminate form!

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1}$$

Apply L'Hôpital's rule.

$$= \frac{1}{1} = \boxed{1}$$

Plug in.

Example!

$$\lim_{x\to 0}\frac{\cos x - 1}{x} \to \frac{1-1}{0} \to \frac{0}{0}$$

Indeterminate form!

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\frac{-\sin x}{1}$$

Apply L'Hôpital's rule.

$$= \frac{-0}{1} = \boxed{0}$$

Plug in.

Some limits produce an **indeterminate form** that cannot be eliminated by factoring. In these cases, **L'Hôpital's rule** is very useful.

These two limits are classic limits that may appear in other situations, such as the limit definition of the derivative for trig functions.

To apply L'Hôpital's rule, you will need to remember the derivatives of $\sin x$ and $\cos x$.

FUNKY limits

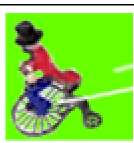
Example!

$$\lim_{x\to 0}\frac{\sin x}{e^x} = \frac{0}{1} = \boxed{0}$$

Not an indeterminate form.

This limit does not meet the criteria for L'Hôpital's rule because it does not produce an indeterminate form. If you tried to use L'Hôpital's rule here, you would get a different answer.

Example!



$$\lim_{x\to\infty}\frac{x^2+3}{x\ln x}$$

racing to infinity

racing to infinity

racing to infinity

Start by plugging in.

$$\lim_{x\to\infty}\frac{x^2+3}{x\ln x} = \lim_{x\to\infty}\frac{2x}{\cancel{x}(1/\cancel{x})+\ln x}$$

Apply L'Hôpital's rule using the product rule $[f(x)\,g(x)]' = f(x)g'(x) + g(x)f'(x)$

$$= \lim_{x\to\infty}\frac{2x}{1+\ln x} \to \frac{\infty}{\infty}$$

Indeterminate form! Needs more work.

$$\lim_{x\to\infty}\frac{2x}{1+\ln x} = \lim_{x\to\infty}\frac{2}{(1/x)}$$

flip

$$= \lim_{x\to\infty} 2x$$

Simplify by multiplying by x/x.

$$= \boxed{\infty}$$

or doesn't exist

In a complicated limit it can be helpful to think about the behavior of specific terms. In this example, the three has a negligible effect. Then the x-squared term in the numerator will overpower $x \ln x$ in the denominator.

After using L'Hôpital's rule once, the limit produces an indeterminate form again. A second application results in an answer.

You can say the limit is infinity, but since that is not a number you can also say it does not exist.