Introduction to the Direct Comparison Test

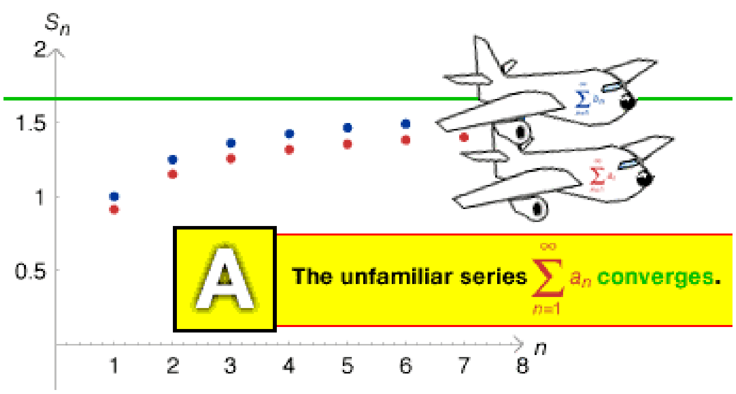
key concepts:

- Use the **direct comparison test** to determine the convergence or divergence of an unfamiliar series by comparing it to a familiar one.

Comparing series

Q If $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, what can you conclude?

Looking at the graph of partial sums makes it easier to see how the series behave.



The direct comparison test: Given $a_n \le b_n$, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

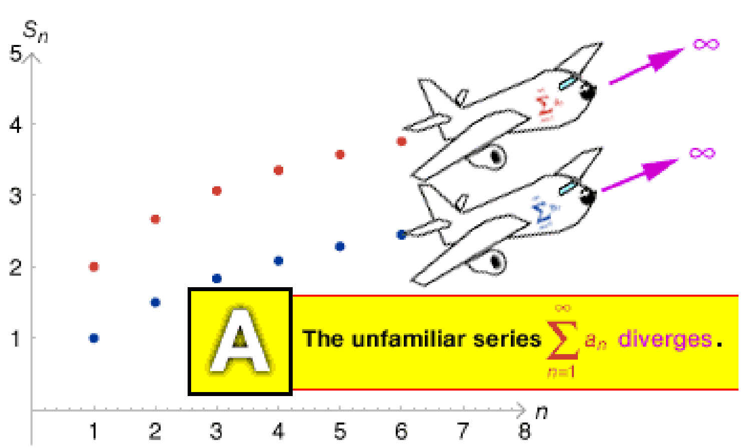
It is possible to determine if a series converges or diverges by comparing it to a different series with which you are familiar. Since the test compares two series by directly examining their terms, it is called the **direct comparison test**.

The first part of the direct comparison test deals with convergence. If you can show that each term of your unknown series is positive and less than or equal to its corresponding term in a known convergent series, then the unknown series must converge.

The reason the series converges is because the terms are positive. That makes the sequence of partial sums an increasing sequence. In addition, the sequence of partial sums is bounded by the sum of the known series. Bounded monotonic sequences must converge. If the sequence of partial sums converges, then the series converges.

Remember, the terms must all be positive for the direct comparison test to work.

Q If $b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, what can you conclude?



The direct comparison test: Given $b_n \le a_n$, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

You can use the direct comparison test to prove that a series diverges too. To do so, just find a divergent series whose terms are less than the terms of your unknown series.

Since you know the lower series diverges and the unknown series is greater, then it must diverge too.

Again, the terms must all be positive for the direct comparison test to work.

Introduction to the Direct Comparison Test

The comparison test in action

Example: Determine whether $\sum_{n=1}^{\infty} \frac{1}{2n^3+1}$ converges or diverges.

This series resembles the p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges because $p = 3$.

In order for the new series to converge, the terms of the p-series must dominate.

Show that: $\frac{1}{2n^3+1} < \frac{1}{n^3}$

$$\frac{1}{2n^3+1} < \frac{1}{2n^3} < \frac{1}{n^3}$$

As the denominator decreases, the fraction increases.

So, $\sum_{n=1}^{\infty} \frac{1}{2n^3+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

The tricky part in using the direct comparison test is finding the right series to compare with.

When looking for a good comparison, start by thinking about which series you are familiar with (like the p-series and the harmonic series) and see if the unknown series resembles any of them.

In this example, the denominator includes the index raised to a constant power. That's very similar to a p-series where $p = 3$.

Now try to determine the relationship between your unknown series and the p-series you want to compare against. Notice that if you reduce the denominator by one the resulting fraction is larger. If you multiply that fraction by two (to cancel the two in the denominator) the resulting fraction is larger. This proves that your p-series is greater than the unknown series.

So you have proven that the unknown series converges. Make sure you include the series you compared it with when you give your answer.

The comparison test in action

Example: Determine whether $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ converges or diverges.

This series is essentially $\sum_{n=1}^{\infty} \frac{n}{n^2}$ or $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is the harmonic series.

In order for the new series to diverge, the terms of the harmonic series must be the smaller ones.

Show that: $\frac{n+1}{n^2} > \frac{1}{n}$

$$\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$$

As the numerator decreases, the fraction decreases.

So, $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

Try to distill the series into its most basic form when looking for a comparison series.

In this example, you could argue that the +1 in the numerator does little to change the series. If you ignored it then you could cancel out one of the n-terms, leaving you with the harmonic series.

Using the harmonic series as your comparison, see if there is a relationship by working backwards from the unknown series.

The series diverges because the terms are greater than the terms of the harmonic series, which you know diverges.

Calculus Lecture Notes

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Unit: Sequences and Series **Module:** The Direct Comparison Test

Using the Comparison Test

key concepts:

- Use the **direct comparison test** to determine the convergence or divergence of an unfamiliar series by comparing it to a familiar one.

Comparing a series to a convergent one

Example!

Consider $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+4)}$.

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+4)} \qquad \sum_{n=1}^{\infty}\frac{1}{n^3}$$

The known series converges, so the unknown series will converge if its terms lie below those of the known one.

Remember! In a p-series, if p is greater than one, then the terms of the series shrink quickly, and the series converges.

$$\frac{1}{n(n+1)(n+4)} < \frac{1}{n\cdot n\cdot n} = \frac{1}{n^3}$$

Reducing the denominator increases the fraction.

So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+4)}$ converges by comparison with the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$.

Does this series converge?

Notice that the series is the product of three factors. Perhaps you can use the **direct comparison test** to compare the series with the ***p*-series** where $p = 3$.

Since adding to the terms of the denominator just makes the fraction smaller, you know that each term of the unknown series is smaller than the corresponding term in the p-series.

So the series converges by the direct comparison test.

Comparing a series to a divergent one

Example!

Consider $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n-2}}$

Q: What is the essence of this series?

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n-2}}$$

A:

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

The known series diverges, so the unknown series will diverge if its terms lie above those of the known one.

Find a term that is smaller than $\dfrac{1}{\sqrt{3n-2}}$.

$$\frac{1}{\sqrt{3n-2}} > \frac{1}{\sqrt{3n}} = \frac{1}{\sqrt{3}\sqrt{n}}$$

Increasing the denominator decreases the fraction.

By the properties of sums: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{3}\sqrt{n}} = \frac{1}{\sqrt{3}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

Therefore $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{3}\sqrt{n}}$ diverges because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ does.

Does this series converge?

The terms of the series are basically one over the square root of the index, with a few constants thrown in. Rewriting the terms in exponential form shows that the comparison series is a p-series. Since $p < 1$, this p-series diverges.

If you can show that the terms of the unknown series are greater than the terms of the comparison series, then the unknown series must diverge.

By making the denominator bigger you make the terms smaller.

You can factor out one over the square root of three. Since the p-series diverges when $p = 1/2$, then the series still diverges when you multiply the series by a constant. So the unknown series diverges by the direct comparison test.

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