Unit: Sequences and Series **Module:** Power Series

Definition of Power Series

key concepts:

- The general form of a **power series** centered at $x = c$ is $\sum_{n=0}^{\infty} a_n (x-c)^n$, where a_n is a sequence of numbers that only depend on n.
- All power series converge for $x = c$. If a power series converges for any other values of x, then it converges for all x-values on an **interval of convergence** centered at $x = c$.

Definition of a power series

Power series are the generalization of infinitely long polynomials.

Power series centered at $x=c$: $\sum_{n=0}^{\infty} a_n (x-c)^n$

Taylor series are a special case of power series where $a_n = \frac{f^{(n)}(c)}{n!}$.

A **power series** is the infinite sum of successive powers of a given variable each accompanied by a coefficient.

Power series are the generalization of Taylor and Maclaurin series.

Consider

EXAMPLE: $1+x+x^2+x^3+\cdots$

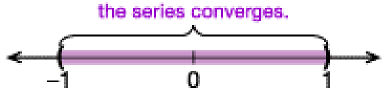
This is a power series with $c=0$ and $a_n=1$.

This is a geometric series with $r=x$ and $a=1$

so $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$.

If $x=3$, $1+3+3^2+3^3+\cdots \neq \frac{1}{1-3}$.

The series will converge to $\frac{1}{1-x}$ only when $|x|<1$, which is the same as $-1<x<1$.



Here is an elementary power series. It is natural to study its convergence.

Notice that this power series can be described as a geometric series. Using the formula for the sum of a geometric series, you can find the sum of this particular series, assuming $|x| < 1$

The series converges for all values of x in the interval (–1, 1).

The interval of convergence

? When does a power series converge?

A1 For $x=c$, $\sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n (c-c)^n$

$$= \sum_{n=0}^{\infty} a_n (0)^n = a_0$$

A2 If the series does converge anywhere besides $x=c$, it will converge for all x-values over an interval centered at $x=c$.

The interval of convergence is the interval of x-values for which the power series converges.

What about the convergence of a general power series?

A power series always converges at $x = c$.

The series may converge for values of x other than $x = c$. If it does, it will converge for a range of values, called the **interval of convergence**. The center of that interval will be c.

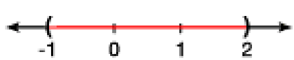
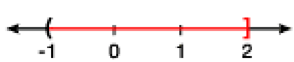
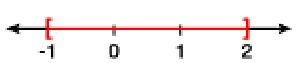
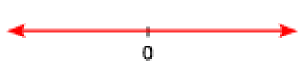
Unit: Sequences and Series **Module:** Power Series

Interval and Radius of Convergence

key concepts:

- The **interval of convergence** of a **power series** is the collection of points for which the series converges.
- The **radius of convergence** of a power series is the distance between the center and either endpoint of the interval of convergence.
- Use the ratio test to find the radius and interval of convergence.

Examples of Intervals

Interval	Explanation	Visual Illustration	Notation
$-1 < x < 2$	Neither endpoint is included in the interval.		$(-1, 2)$
$-1 \le x < 2$	The left endpoint is included in the interval.		$[-1, 2)$
$-1 < x \le 2$	The right endpoint is included in the interval.		$(-1, 2]$
$-1 \le x \le 2$	Both endpoints are included in the interval.		$[-1, 2]$
$-\infty < x < \infty$	All real numbers are included in the interval.		$(-\infty, \infty)$

There are four kinds of intervals involving endpoints that are finite only. If an endpoint is not included in the interval, a parenthesis is used. If it is included, a square bracket is used.

For intervals with endpoints that are infinite, parentheses are always used. Infinity is not a number that can be included in an interval.

Finding the interval and radius of convergence

Example: Consider $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

Use the ratio test to find the radius and interval of convergence.

$$\lim_{n\to\infty}\left|\frac{\cancel{x^n}\,x^{1}}{\cancel{x^n}}\right| = \lim_{n\to\infty}|x|$$

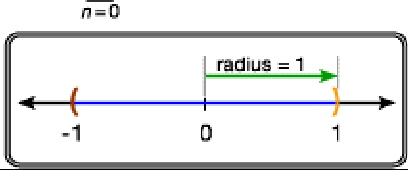
$$= |x| = \rho$$

The ratio test states that the series converges for $\rho < 1$.

Set $|x| < 1$, then the series converges.

If $x = 1$ $\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1^2 + \cdots$, the series diverges.

If $x = -1$ $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - \ldots$, the series diverges.



This is a geometric series. It converges if $|x| < 1$. The series—and therefore the interval of convergence—is centered about $x = 0$.

You can always use the ratio test to determine the **interval of convergence** of a **power series**.

For this series, ρ is equal to $|x|$. If $|x| < 1$, then the series converges. So the series converges on the interval $(–1, 1)$.

Don't forget to study the endpoints. For $x = 1$, the sum is infinite. For $x = –1$, the sum alternates between zero and one. So the series diverges for both endpoints.

The interval of convergence is just $(–1, 1)$. The center of this interval is zero. The **radius of convergence**, which is the distance from the center to either endpoint, is one.

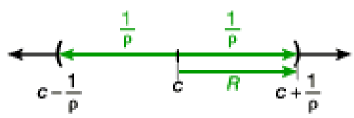
Finding the Interval and Radius of Convergence – Part One

key concepts:

- Use the ratio test to find the **radius of convergence** and **interval of convergence** of a power series.
- The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

Deriving the interval and radius of convergence

Power Series generic brand $\sum_{n=0}^{\infty} a_n (x-c)^n$



Use the ratio test to find the interval of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{\cancel{n+1}}}{a_n\cancel{(x-c)^n}}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)}{a_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x-c|$$

$$= \rho|x-c|$$

The ratio test states that the series converges when $\rho|x-c| < 1$, so $|x-c| < \frac{1}{\rho}$. radius of convergence

The radius of convergence is $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$.

Suppose you wanted a general formula for the **interval of convergence** and **radius of convergence** of any power series.

Use the ratio test on a generic power series.

Set up the limit. A lot of the $(x-c)$ factors cancel.

Remember, the limit is in term of n, so it considers x to be a constant. After you factor out $|x-c|$, You can call the remaining limit ρ (rho).

The ratio test states that the power series converges for all x that make the result less than one. A little algebraic manipulation gives you the general radius of convergence, $1/\rho$.

Here's another way to express the radius of convergence.

Finding the interval and radius of convergence

example: $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Q What are the radius and interval of convergence?

Use the ratio test!

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!\,x^{\cancel{n+1}}}{(n+1)!\,\cancel{x^n}}\right|$$

Set $\lim_{n\to\infty}\frac{1}{n+1}|x| < 1$ $\qquad \lim_{n\to\infty}\frac{1}{n+1} = 0$

$0|x| < 1$

$0 < 1$ ALWAYS!

A The radius of convergence is $\lim_{n\to\infty}\frac{1}{\frac{1}{n+1}} = \lim_{n\to\infty} n+1 = \infty$.

Take another look at this familiar power series.

You can use the formula you just derived or you can start from the ratio test again.

Set up the limit, then invert and multiply. Cancel where possible.

When you factor out $|x|$, all that's left is a limit that is equal to zero.

Zero times any real number is zero.

Zero is always less than one. So this series converges for all values of x.

Using the formula for radius of convergence produces an infinite radius. The interval of convergence is $(-\infty, \infty)$

Finding the Interval and Radius of Convergence – Part One

Finding the interval and radius of convergence

example: $\sum_{n=1}^{\infty} \frac{x^n}{n5^n}$

Q What are the radius and interval of convergence?

Use the ratio test!

You might memorize the formula for finding the radius and interval of convergence of a series like this one. Still, it's a good idea to understand how it can be derived from the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{[(n+1)5^{n+1}]}}{\frac{x^n}{n5^n}}\right| = \lim_{n\to\infty}\left|\frac{n\cancel{5^n}\,\cancel{x^{n+1}}}{(n+1)\cancel{5^{n+1}}\,\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{nx}{5(n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{5(n+1)}\right||x|$$

$$= \frac{1}{5}|x| \quad \text{Use L'Hôpital's rule.}$$

Set up the limit of the ratio. Then invert and multiply.

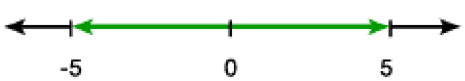
Cancel common factors.

Factor out $|x|$. Evaluating the remainder of the limit produces 1/5.

Set $\frac{1}{5}|x| < 1$

$|x| < 5$ radius of convergence

The series converges for those values of x that make the limit less than one. When you solve this inequality, you see that the radius of convergence is five.



Since the series involves x^n, the center is zero.

Remember that the ratio test is inconclusive when the limit equals one. That means that you have to check the endpoints using some other test.

Check The Endpoints!

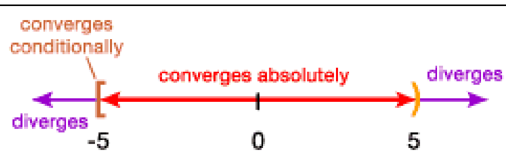
If $x = 5$, $\sum_{n=1}^{\infty} \frac{x^n}{n5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series and it diverges.

When you plug in $x = 5$, the series reduces to the harmonic series, which diverges. So that endpoint is not in the interval of convergence.

Check The Endpoints!

If $x = -5$, $\sum_{n=1}^{\infty} \frac{x^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is the alternating harmonic series and it converges conditionally.

When you plug in $x = -5$, the series reduces to the alternating harmonic series, which converges conditionally. That endpoint is in the interval of convergence.



A The radius of convergence is 5.
The interval of convergence is [-5,5).

Now you have a graphical representation of the interval of convergence of the power series.

Finding the Interval and Radius of Convergence – Part Two

key concepts:

- Use the ratio test to find the **radius of convergence** and **interval of convergence** of a power series.

Examples finding the radius and interval of convergence

example: $\sum_{n=1}^{\infty} \frac{x^n}{n}$

What are the x-values for which this power series converges?

Notice that the interval of convergence will be centered at $x = 0$, since all the terms of the series are zero at that point.

absolute convergence

$-1 \quad 0 \quad 1$

Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{nx^{\cancel{n+1}}}{(n+1)\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\frac{n}{n+1}|x| = |x| \qquad \text{Set } |x| < 1$$

When trying to determine the **interval of convergence** and **radius of convergence** of a power series, it is a good idea to identify the center first. Since $x = 0$ makes all the terms be equal to zero, it's the center.

Use the ratio test to determine the convergence of the power series.

Set up the limit of the ratio, then invert and multiply. Cancel where possible.

Factor out $|x|$. The remaining limit is one, so the result is $|x|$. The radius of convergence is one.

Check the endpoints!

If $x = 1$, $\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the harmonic series, and it diverges.

If $x = -1$, $\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which is the alternating harmonic series, and it converges conditionally.

You cannot use the ratio test to check the endpoints, because it is inconclusive. Instead you have to plug in the endpoints and use another test.

For $x = 1$, the power series equals the harmonic series, which diverges.

For $x = -1$, the power series equals the alternating harmonic series, which converges conditionally.

Therefore the interval of convergence is $[-1, 1)$.

example: $\sum_{n=1}^{\infty} n^n(x-1)^n$

Find the interval of convergence.

Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}(x-1)^{\cancel{n+1}}}{n^n\cancel{(x-1)^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}|x-1|$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)^n}{n^n}|x-1|$$

$$= \lim_{n\to\infty}(n+1)\left(\frac{n+1}{n}\right)^n|x-1|$$

$$= \lim_{n\to\infty}(n+1)\,e\,|x-1|$$

$$= \infty \qquad \lim_{n\to\infty}(n+1) = \infty$$

Notice that the center of the interval of convergence for this power series will be $x = 1$.

Use the ratio test. Set up the limit of the ratio and cancel where possible. Factor out $|x - 1|$.

Factor the numerator.

Notice that the middle part of this limit approaches e.

As n approaches infinity, this limit is infinite. It is never less than one! So what does that say about the interval of convergence?

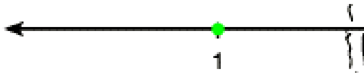
Finding the Interval and Radius of Convergence – Part Two

example: $\sum_{n=1}^{\infty} n^n(x-1)^n$

Find the interval of convergence.

For $x \neq 1$, the series diverges.

The interval of convergence is the point $x = 1$.



The radius of convergence is zero.

Since the ratio test produced $\rho = \infty$…

…this power series diverges for all x-values except one.

The interval of convergence collapses down to a single point. So the radius of convergence is effectively zero.

This weird interval results from the fact that the ratio test limit blows up to infinity.

Examples finding the radius and interval of convergence

example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

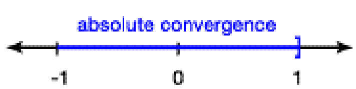
What are the radius and interval of convergence?

The radius of convergence is one.

Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{n^2 x^{\cancel{n+1}}}{(n+1)^2 \cancel{x^n}}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x| = |x|$$

Set $|x| < 1$

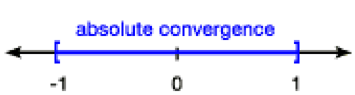


Check the endpoints!

If $x = 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is the p-series with $p = 2 > 1$, and it converges.

Check the endpoints!

If $x = -1$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges absolutely because if you apply the absolute value, it becomes a p-series with $p = 2 > 1$.



What is the center of this power series? The value $x = 0$ makes all the terms of this power series equal to zero, so it's the center.

Apply the ratio test by setting up the limit and canceling where possible.

Factor out $|x|$. The remaining limit is one, leaving just $|x|$.

Set the result less than one. By the ratio test, the radius of convergence is one.

Make sure to check the endpoints.

When you plug in $x = 1$, you get a p-series with $p = 2$, which converges absolutely.

When you plug in $x = -1$, you get an alternating series. If you take the absolute value of the terms, you another p-series with $p = 2$, which converges absolutely.

Therefore the interval of convergence is [–1, 1]. The power series converges absolutely for all values in this interval.

Finding the Interval and Radius of Convergence – Part Three

key concepts:

- Use the ratio test to find the **radius of convergence** and **interval of convergence** of a power series.

Finding the radius and interval of convergence

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$

What are the radius and interval of convergence?

Use the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n x^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\sqrt{n}\, x^{\cancel{n+1}}}{(-1)^n \sqrt{n+1}\, \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}\,|x|$$

$$= |x|$$

Set $|x| < 1$

BE CAREFUL!
Write slowly and legibly.

Apply the ratio test to determine the radius and interval of convergence.

Set up the limit, invert and multiply, and cancel where possible.

The –1 factors are eliminated by taking the absolute value.

The limit of the expression under the square root is one. The square root of one is one, so you are left with $|x|$. Find the values that make it less than one..

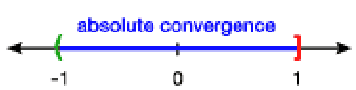
The limit you started with is huge and has lots of exponents. Write large and take your time. That's the best way to avoid errors.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$

What are the radius and interval of convergence?

The radius of convergence is one.

Set $|x| < 1$



Check the Endpoints

If $x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which is an alternating series that converges conditionally by the alternating series test.

If $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a p-series with $p = 0.5 < 1$ and it diverges.

By the ratio test, the radius of convergence of this power series is one. It is centered about $x = 0$.

The power series converges absolutely inside the interval, but you still need to study the endpoints.

For $x = 1$, the series becomes an alternating series. Since the non-alternating parts are decreasing and approaching zero, it converges conditionally. It does not converge absolutely because the absolute value produces a p-series with $p = 1/2$.

For $x = -1$, the series has two $(-1)^n$ factors that cancel. That leaves a p-series with $p = 1/2$, which diverges.

The interval of convergence is $(-1, 1]$.

Finding the Interval and Radius of Convergence – Part Three

Finding the radius and interval of convergence

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n2^n}$ What are the radius and interval of convergence?

Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x-3)^{n+1}}{(n+1)2^{n+1}}}{\dfrac{(-1)^n(x-3)^n}{(n)2^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n)\cancel{2^n}(x-3)^{\cancel{n+1}}}{(-1)^n(n+1)2^{\cancel{n+1}}\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{n}{2(n+1)}|x-3|$$

$$= \frac{1}{2}|x-3|$$

Set $\frac{1}{2}|x-3| < 1$

$$\boxed{|x-3| < 2}$$

The value that makes all the terms of this power series zero is $x = 3$. This is the center.

This is another complicated series. Make sure to write big and go slowly to avoid making mistakes.

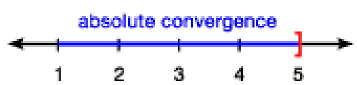
Set up the limit, then invert and multiply.

Cancel where possible.

The –1 factors disappear with the absolute value.

For the power series to converge, the limit must be less than one.

To find the radius of convergence, you need to multiply both sides of the inequality by two. The radius is two.



Check the Endpoints

If $x = 5$, $\sum_{n=1}^{\infty}\frac{(-1)^n(x-3)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(5-3)^n}{n2^n}$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n(2)^n}{n2^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

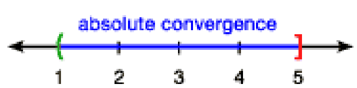
, which is the alternating harmonic series and converges conditionally.

The interval of convergence is centered about $x = 3$. Inside the interval you have absolute convergence.

As usual, you have to check the endpoints. In this case the endpoints are $x = 1$ and $x = 5$.

Plug in $x = 5$ and cancel where possible.

The result converges conditionally.



Check the Endpoints

If $x = 1$, $\sum_{n=1}^{\infty}\frac{(-1)^n(x-3)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(1-3)^n}{n2^n}$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n(-2)^n}{n2^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n(2)^n}{n2^n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}$$

, which is the harmonic series and diverges.

Plug in $x = 1$.

Factor $(–2)^n$ into $(–1)^n$ and $(2)^n$. Cancel the $(2)^n$ factors.

The result is the harmonic series, which diverges.

Therefore the interval of convergence is (1, 5].