**Unit:** Sequences and Series **Module:** The Integral Test

# Introduction to the Integral Test

**key concepts:**

- The sum of a series can be associated with an area. If you can show that the area is finite, then the series converges.
- The **integral test**:

  Suppose $f$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$.

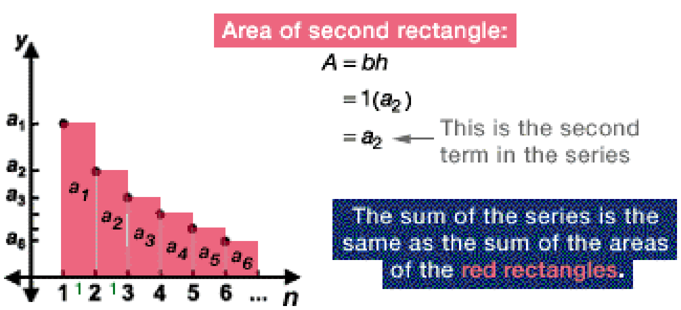
  If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

## Expressing a series as an area

Consider $\sum_{n=1}^{\infty} a_n$. The sequence is positive and decreasing.

Suppose $a_n > 0$ and $a_1 > a_2 > a_3 > \ldots$




The first test you will learn to check for convergence of a series is the **integral test**.

The integral test ties series to definite integrals through the analysis of area.

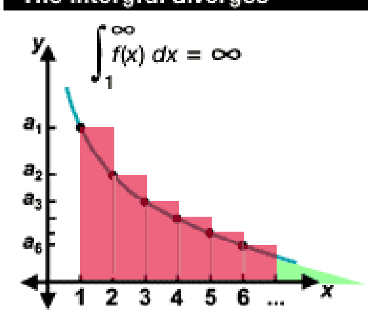
For the integral test to work, you must have a positive, decreasing series.

Notice that the sum of the series can be represented as the area of the rectangles with height equal to the term and width equal to one.

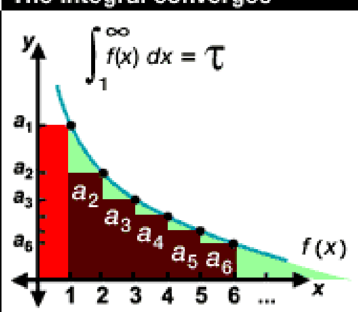
## Convergent integral, convergent series

Consider Suppose $a_n > 0$ and $a_1 > a_2 > a_3 > \ldots$ $\sum_{n=1}^{\infty} a_n$.

**The intergral diverges**




**The integral converges**




One application of the definite integral is to find the area under a curve. If you can find a function that produces the terms of the series when evaluated along the natural numbers, then you can take the definite integral of that function and compare the two areas.

If the definite integral evaluated from one to infinity (which is an **improper integral**) diverges, then the series also diverges. Notice that in the diagram the area of the series is greater than the area under the curve. Since the area under the curve is infinite, the series would have to be infinite too.

However, if the definite integral evaluated from one to infinity converges, then the series converges. Since the area under the curve is finite, the value of the series from the second through the final term is also finite. Adding in the first term still leaves you with a finite value.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# Examples of the Integral Test

key concepts:

- The **integral test**:

  Suppose $f$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$.

  If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

- The integral test does not give you the value of the series. The definite integral used to test the series is not equal to the infinite series.

### *The harmonic series*

Example!

Consider $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

Use the integral test.

$$\int_1^\infty \frac{1}{x}dx = \lim_{a\to\infty}\int_1^a \frac{1}{x}dx$$

Use a limit to evaluate the improper integral.

$$= \lim_{a\to\infty} \ln|x|\Big|_1^a$$

$$= \lim_{a\to\infty} \ln|a| - 0$$

ln|1| is zero.

$$= \lim_{a\to\infty} \ln|a|$$

As $a$ gets really large, so does ln|a|.

$$= \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Even though the terms of the harmonic series approach zero, the series diverges.

Some series are so common that they have names. The **harmonic series** is produced by the reciprocal function. Does the harmonic series converge?

To check for convergence with the **integral test**, the function must be positive, continuous, and decreasing for all $x$ greater than or equal to one. All these conditions are true for the harmonic series, so you can use the integral test.

The integral of $1/x$ is $\ln x$.

Notice that the **improper integral** diverges. Therefore the harmonic series diverges by the integral test.

The “quicky test” would have been inconclusive here since the terms approach zero.

### *A convergent series*

Example!

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Use the integral test.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{a\to\infty}\int_1^a \frac{1}{x^2}dx$$

Use a limit to evaluate the improper integral.

$$= \lim_{a\to\infty} \frac{-1}{x}\Big|_1^a$$

$$= \lim_{a\to\infty}\left(\frac{-1}{a} - \frac{-1}{1}\right)$$

As $a$ gets really large, the first term goes to zero.

$$= 0 - (-1)$$

$$= 1 \longleftarrow \text{FINITE!!}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Even though $\int_1^\infty \frac{1}{x^2}dx = 1$, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Now consider an example very similar to the harmonic series.

The function is positive, continuous, and decreasing for all $x$ greater than or equal to one, so you can use the integral test.

The improper integral converges to one. So the series must converge. But the series does not have to converge to one.

In fact, the series actually converges to $\pi^2/6$.



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# Using the Integral Test

## key concepts:

- The **integral test**:

  Suppose *f* is positive, continuous, and decreasing for $x \ge 1$ and $a_n = f(n)$.

  If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

- One way to verify that the function *f* is decreasing is to take its derivative. If the derivative is always negative for $x \ge 1$, then the function is decreasing.

### An exponential series

Consider $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Q Does this series converge or diverge?

The terms meet the necessary conditions. You can use the integral test.

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \lim_{a\to\infty} \int_1^{a} \frac{x}{e^{x^2}}\,dx$$

Replace $e^{x^2}$ with $u$.

$$\int \frac{1}{2u}\left(\frac{1}{u}\right)du = \frac{1}{2}\int \frac{1}{u^2}\,du$$

$$= \frac{-1}{2}\cdot\frac{1}{u} + C$$

$$= \frac{-1}{2u} + C$$

$$\lim_{a\to\infty} \left.\frac{-1}{2e^{x^2}}\right|_1^a = \lim_{a\to\infty}\left(\frac{-1}{2e^{a^2}} - \frac{-1}{2e^{1^2}}\right)$$

$$= 0 - \left(\frac{-1}{2e}\right)$$

$$= \boxed{\frac{1}{2e}}$$

Try using the **integral test** to check for convergence. The first step of the integral test is to make sure that the function is positive, continuous, and decreasing for all *x* greater than or equal to one. You can verify that this function is decreasing by taking the first derivative.

Now that you have satisfied the initial conditions of the integral test, you can evaluate the improper integral.

Notice that the improper integral converges. Therefore the series converges.

**Remember:** The integral and the series do not have to converge to the same value.

### Hidden signs

**Example!**

Consider $\sum_{n=1}^{\infty} \frac{\sin n}{e^n}$

The terms approach zero since the denominator dominates.

Q Is the integral test appropriate here?

A No! The terms are not always positive. Sine takes on both positive and negative values.

How could you check for convergence on this series?

At first glance, the series looks like a good candidate for the integral test. But stop and think about the sine function for a moment.

Sine is not a decreasing function. It is not even **monotonic**. It shifts from positive to negative.

Since the function is not decreasing, then the integral test does not work here.

Be careful! It is very easy to use the integral test inappropriately if you do not make sure the function is positive, decreasing, and continuous first.



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# Defining $p$-Series

key concepts:

- The **integral test**:

  Suppose $f$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$.

  If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

  If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.

- A ***p*-series** is a series of the form $\sum_{n=1}^\infty \frac{1}{n^p}$, where $p$ is a positive number.

| | |
|---|---|
| ***The general form*** <br> Let $p > 0$ be a fixed number. <br> $p$-Series <br> **p-series:** $\sum_{n=1}^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ <br> The **exponent** remains fixed. <br> **geometric series:** $\sum_{n=1}^\infty r^n = r^1 + r^2 + r^3 + \ldots$ <br> The **base** remains fixed. | Another series you will see a lot of is the ***p*-series**. <br><br> In a geometric series, the exponent grows and the base remains fixed. <br><br> In a $p$-series, it is the other way around. The exponent remains the same while the base grows. |
| ***Convergence of a p-series*** <br> Example! <br> harmonic series <br> $\sum_{n=1}^\infty \frac{1}{n}$ is a $p$-series with $p = 1$. <br> diverges <br> $p$-Series <br> You know that when $p = 1$, the $p$-series diverges. | The most basic $p$-series is the series for $p = 1$. This series is also known as the **harmonic series**. <br><br> You have already seen that the harmonic series diverges. But what happens if you use a different number for $p$? |
| **Case 1: $p > 1$** <br> $\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx$ <br> $= \lim_{a\to\infty} \int_1^a x^{-p}dx$ <br> $= \lim_{a\to\infty} \frac{x^{(-p+1)}}{-p+1}\Bigg\rvert_1^a$ <br> $= \lim_{a\to\infty} \frac{a^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}$ <br> $= -\frac{1}{-p+1}$ <br> $= \frac{1}{p-1}$ **Converges!** <br><br> **Case 2: $p \leq 1$** <br> $\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx$ <br> $= \lim_{a\to\infty} \int_1^a x^{-p}dx$ <br> $= \lim_{a\to\infty} \frac{x^{(-p+1)}}{-p+1}\Bigg\rvert_1^a$ <br> $= \lim_{a\to\infty} \frac{a^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}$ <br> $= \infty$ **Diverges!** | It turns out that if $p$ is greater than one then the $p$-series converges. <br><br> You can verify this result with the **integral test**. <br><br> Since $p$ is greater than one, then $(-p + 1)$ is less than zero. When you take the limit of the expression as $a$ approaches zero, the improper integral converges. <br><br> But when $p$ is less than one, then $(-p + 1)$ is greater than zero. Taking the limit now gives you a divergent improper integral. <br><br> So to determine if a $p$-series converges, just check the value of $p$. |


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