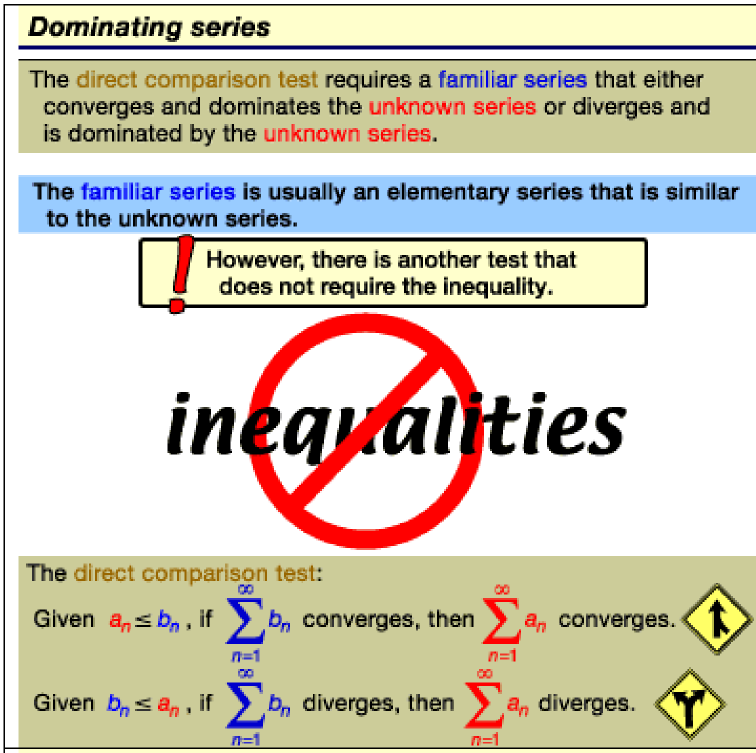
Introduction to the Limit Comparison Test

key concepts:

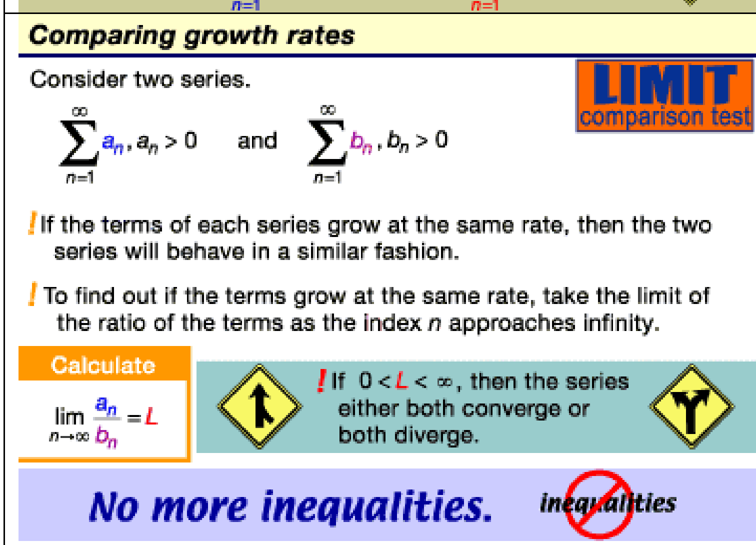
- The **limit comparison test**: Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, with $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then either both series converge or both diverge.



The **direct comparison test** enables you to determine if a series is convergent or divergent by comparing the unknown series to a known series.

It isn't always easy to find a familiar series that bounds the unknown series. The presence of inequalities makes the direct comparison test somewhat impractical for analyzing many series.

It would be good to find a series test that didn't require you to compare two series together with an inequality.



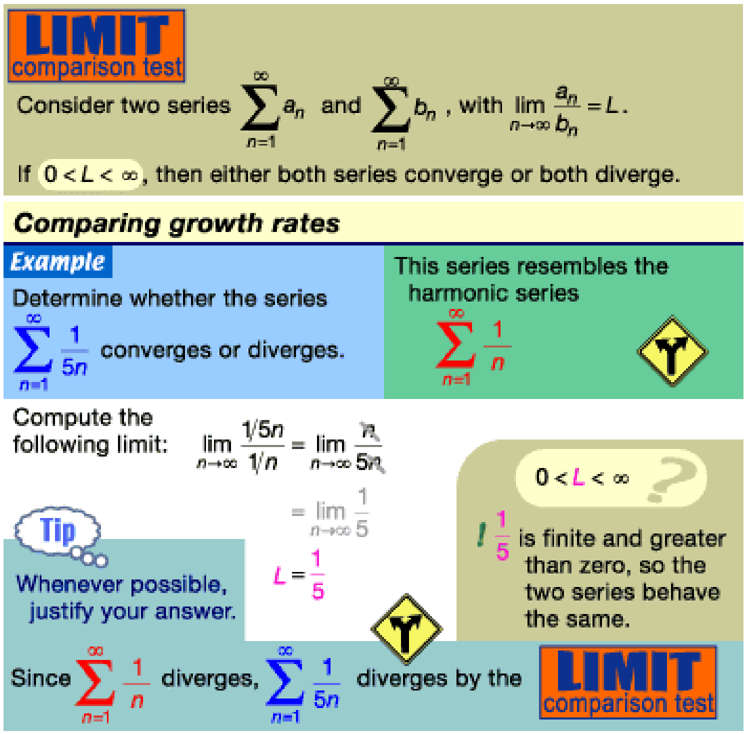
Consider two arbitrary series whose terms are positive.

You can actually learn about the behavior of the two series by looking at the limit of the ratio of their terms as the index approaches infinity.

If the limit is positive and finite, then the series behave the same. Either they both converge or they both diverge.

The limit comparison test is easier to use since you don't have to worry about boundedness.

Introduction to the Limit Comparison Test



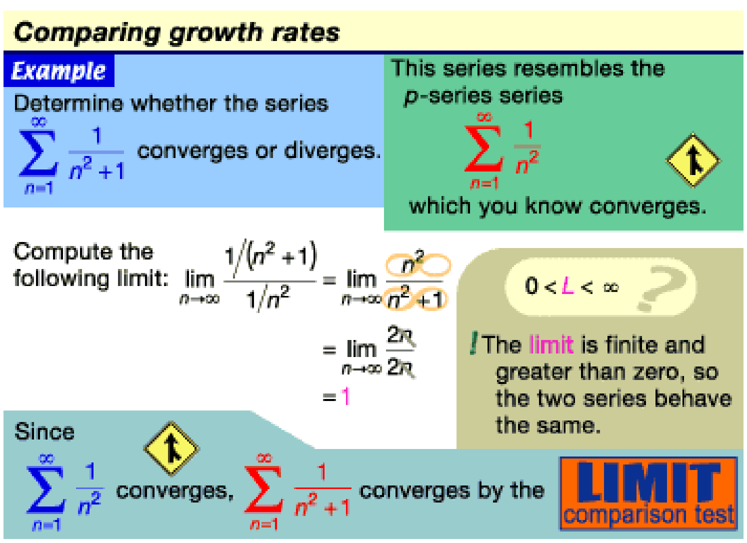
Here is a formal description of the limit comparison test. All the definition says is that if the limit of the ratio of terms is finite and positive then the two series behave the same way.

Here is a basic example. Does this series converge or diverge? You can compare the series with the **harmonic series**.

The limit of the ratio of the terms of the series is finite and positive. Therefore the two series behave the same by the limit comparison test.

Since the harmonic series diverges, this series diverges too.

Notice that you could have proven this series was divergent by factoring out the constant and showing that the series was a constant times the harmonic series.



Consider this series. You can't prove this series is convergent or divergent by factoring out any constants.

The series resembles the p-series with $p = 2$. So use that series when applying the limit comparison test.

The limit of the terms produces an indeterminate form. L'Hôpital's rule simplifies the limit.

The limit of the ratio is finite and positive. So the two series behave the same. Since the p-series converges, the unknown series converges too.

See how much easier it is to prove the series are convergent when you don't have to mess with the inequalities?

Unit: Sequences and Series **Module:** The Limit Comparison Test

Using the Limit Comparison Test

key concepts:

- The **limit comparison test**: Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, with $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then either both series converge or both diverge.
- If the limit of the ratio of the nth terms is zero or does not exist, then the limit comparison test is inconclusive.

A bad choice for comparison

EXAMPLE Determine whether $\sum_{n=1}^{\infty} \frac{n+1}{3n^3-1}$ converges or diverges.

Compare with the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges because it is a p-series with $p = 3$.

$$\lim_{n\to\infty} \frac{(n+1)/(3n^3-1)}{1/n^3} = \lim_{n\to\infty} \frac{n^3(n+1)}{3n^3-1}$$ Invert and multiply.

$$= \lim_{n\to\infty} \frac{n^4+n^3}{3n^3-1} \to \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{4n^3+3n^2}{9n^2}$$ Use L'Hôpital's rule.

$$= \lim_{n\to\infty} \frac{4n+3}{9}$$ Divide by n^2.

$$= \infty$$

$0 < L < \infty$?

! Since the limit does not exist, the limit comparison test is inconclusive.

You have to be careful when you pick a series for comparison when using the **limit comparison test**. Some choices reveal more than others.

Here a comparison is made with the p-series where $p = 3$.

Using L'Hôpital's rule shows that the limit of the ratio is undefined.

Since the limit is undefined, the limit comparison test is inconclusive. That does not mean that the series behaves differently. It means that the test didn't tell you anything about the behavior of the series.

A better choice for comparison

EXAMPLE Determine whether $\sum_{n=1}^{\infty} \frac{n+1}{3n^3-1}$ converges or diverges.

Compare with $\sum_{n=1}^{\infty} \frac{n}{n^3}$

The numerator grows like n while the denominator grows like n^3.

$$\lim_{n\to\infty} \frac{(n+1)/(3n^3-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2(n+1)}{3n^3-1}$$ Invert and multiply.

$$= \lim_{n\to\infty} \frac{n^3+n^2}{3n^3-1} \to \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{3n^2+2n}{9n^2} \to \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{6n+2}{18n} \to \frac{\infty}{\infty}$$

$$= \frac{1}{3}$$ Use L'Hôpital's rule again and simplify.

$$\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ This is also a p-series with $p > 1$, so it converges.

Reconsider the series that was compared to the unknown series. Notice that the familiar series above doesn't really look like the unknown series. The unknown series has an n-term in the numerator. So try putting one into the familiar series. Now the familiar series is a p-series where $p = 2$.

This time the limit is positive and finite.

Since the limit is positive and finite, the unknown series behaves the same way as the familiar series. Since the familiar series is a p-series with $p > 1$ then it converges. Therefore the unknown series converges as well.

Unit: Sequences and Series **Module:** The Limit Comparison Test

Inverting the Series in the Limit Comparison Test

key concepts:

- The **limit comparison test**: Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, with $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then either both series converge or both diverge.
- If the limit of the ratio of the two terms is zero or does not exist then the limit comparison test is inconclusive.
- You can put the unknown series in either the numerator or denominator when using the limit comparison test.

Changing roles

EXAMPLE: Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^4 + 2\sqrt{n}}$ converges or diverges.

LIMIT comparison test

Compare with the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges because it is a p-series with $p > 1$.

$$\lim_{n\to\infty} \frac{1/\left(n^4 + 2\sqrt{n}\right)}{1/n^4} = \lim_{n\to\infty} \frac{n^4}{n^4 + 2\sqrt{n}}$$ Table the original limit.

$$\lim_{n\to\infty} \frac{1/n^4}{1/\left(n^4 + 2\sqrt{n}\right)} = \lim_{n\to\infty} \frac{n^4 + 2\sqrt{n}}{n^4}$$ Invert and multiply.

$$= \lim_{n\to\infty} \left(\frac{\cancel{n^4}}{\cancel{n^4}} + \frac{2\sqrt{n}}{n^4}\right)$$ Separate the fractions.

$$= 1 + \lim_{n\to\infty} \frac{2n^{1/2}}{n^{8/2}}$$ Convert to fractional exponents.

$$= 1 + \lim_{n\to\infty} \frac{2}{n^{7/2}}$$ Cancel $n^{1/2}$.

$$= 1 + 0$$

$$= 1$$

Notice that when using the limit comparison test, sometimes the limit of the ratio of terms can be a little tricky to solve.

But since the limit comparison test doesn't specify which term must be where, you can set up the ratio so that it is easier to solve. In this example, the limit is easier to solve when the familiar series is in the numerator instead of the denominator.

So the limit of the ratio is finite and positive. Therefore the unknown series converges by comparison with the p-series where $p = 4$.

Combining radicals and transcendental functions

EXAMPLE: Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 2\ln n}$ converges or diverges.

LIMIT comparison test

Compare the series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges because it is a p-series with $p \le 1$. $\frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$

$$\lim_{n\to\infty} \frac{1/\sqrt{n}}{1/\left(\sqrt{n} + 2\ln n\right)} = \lim_{n\to\infty} \frac{\sqrt{n} + 2\ln n}{\sqrt{n}}$$ Invert and multiply.

$$= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n}} + \frac{2\ln n}{\sqrt{n}}\right)$$ Separate the fractions.

$$= 1 + \lim_{n\to\infty} \frac{2\ln n}{\sqrt{n}} \to \frac{\infty}{\infty}$$

$$= 1 + \lim_{n\to\infty} \frac{2/n}{1/\left(2\sqrt{n}\right)}$$ Use L'Hôpital's rule.

$$= 1 + \lim_{n\to\infty} \frac{4\sqrt{n}}{n}$$ Invert and multiply.

$$= 1 + \lim_{n\to\infty} \frac{4}{\sqrt{n}}$$ Simplify.

$$= 1 + 0 = 1$$

In this example, the familiar series was placed in the numerator. Because the series involves a reciprocal function, the simpler term will move into the denominator when you invert and multiply.

Since the denominator is now a single term, you can break the more complicated term into two pieces and find the limits of each piece separately.

The limit is finite and positive, so the unknown series diverges by comparison with the p-series where $p = 1/2$.