Unit: Techniques of Integration **Module:** Introduction to Integration by Partial Fractions

Finding Partial Fraction Decompositions

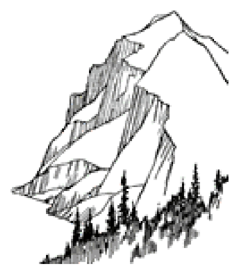
key concepts:

- Decomposing a **rational expression** into **partial fractions** produces an equivalent expression that is usually easier to integrate.
- Break up difficult rational functions by following these two steps:
  1. Write the function as the sum of fractions with unknown numerators.
  2. Solve for the unknown numerators.

An integration challenge

Integrate $\int \frac{-1}{x^2 - x}\,dx.$

$$\frac{-1}{x^2 - x} = \frac{-1}{x(x-1)}\,!$$



Here is a tricky integral.

Notice that the integral can't be solved with any of the techniques you have learned so far. There is no way to make a *u*-substitution, and the integral can't be broken up by trig identities since it doesn't involve trig expressions.

In fact, the only thing you can do to the integrand is factor out an x.

This integrand is an example of a **rational expression**. In general, rational expressions can't be integrated by *u*-substitution. Is there another way to integrate a rational expression?

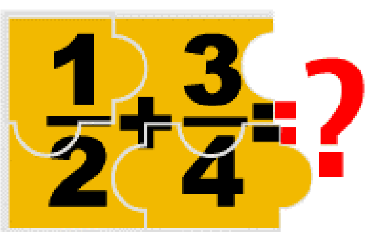
A jigsaw puzzle

Q Is it true that $\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$?

A Yes. You can verify this by finding a common denominator.

$$\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

$$\frac{2}{4} + \frac{3}{4} = \frac{5}{4}$$



When you were young, you started with the left side.

Think back to when you learned to add fractions.

When you first worked this type of problem you were given two fractions and asked to combine them together. By finding a common denominator, you were able to make the two fractions into one.

But suppose you were given the answer first. How could you go backwards and find the two fractions? Going backwards like this is called decomposing the fraction.

Algebraic fractions

Q Why would you want to decompose fractions?

Integrate $\int \frac{-1}{x^2 - x}\,dx.$

$$\frac{-1}{x^2 - x} = \frac{-1}{x(x-1)}\,!$$

Combine $\frac{1}{x} - \frac{1}{x-1} = \frac{(x-1)}{x(x-1)} - \frac{x}{x(x-1)}$

$$= \frac{x-1-x}{x(x-1)}$$

$$= \frac{-1}{x(x-1)}$$

This is the integrand of the original challenge problem!

Partial fractions are simpler fractions which sum to form a given rational function.

It does seem silly to take an answer and try to work backwards and find the question. But consider this example.

Notice the expression in the far left. By finding the common denominator and adding like terms you turn those two fractions into a more complicated fraction. Notice how the original expression is in many ways simpler than the result?

It turns out that the original expression is equal to the complicated integrand in the example above!

The fractions that make up the original expression are called **partial fractions**.

Unit: Techniques of Integration **Module:** Introduction to Integration by Partial Fractions

Finding Partial Fraction Decompositions

So $\int \frac{-1}{x(x-1)}dx = \int \left(\frac{1}{x} - \frac{1}{x-1}\right)dx$, which you know how to integrate.

$$\int \frac{-1}{x^2 - x}dx = \int \left(\frac{1}{x} - \frac{1}{x-1}\right)dx$$

$$\boxed{= \ln|x| - \ln|x-1| + C}$$

A Decomposing a fraction gives two functions which are more elementary and therefore can be integrated.

Since the decomposition is equal to the original expression, you can substitute one for the other.

Notice that the partial fraction decomposition of the original integrand is much easier to integrate than the original rational expression.

Steps for partial fraction decomposition

Q How do you decompose a function into partial fractions?

A 1. Write the function as the sum of two fractions with unknown numerators.

$$\frac{-1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$= \frac{A(x-1)}{x(x-1)} + \frac{Bx}{x(x-1)}$$

$$= \frac{Ax - A + Bx}{x(x-1)}$$

$$\boxed{\frac{-1}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}}$$

It's an IDENTITY

2. Solve for unknown numerators.

Solve by identifying the parts of each expression.

$A + B = 0$ and $-A = -1$

so

$A = 1$ and $B = -1$

It is useful to find **partial fraction decompositions** when integrating rational expressions. But how do you find a partial fraction decomposition?

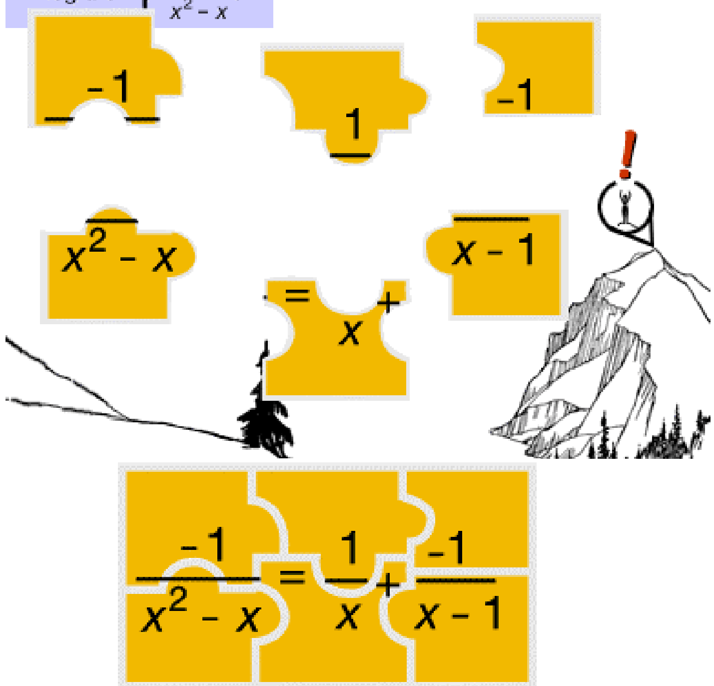
Start by breaking the factored denominator into different terms, each with an unknown numerator.

Now combine the terms. Here you take a common denominator and add the two numerators together.

Notice that the two expressions are equal. This equation is an **identity**.

Now that you have built the identity, set each piece of the numerator of the first expression equal to its corresponding piece of the second expression and solve for A and B.

Integrate $\int \frac{-1}{x^2 - x}dx$.



By taking the original expression and breaking it up. . .

you found a new expression that is easier to work with.

This is the idea of the partial fraction decomposition technique.

Unit: Techniques of Integration **Module:** An Introduction to Integration by Partial Fractions

Partial Fractions

key concepts:

- Decomposing a **rational expression** into **partial fractions** produces an equivalent expression that is usually easier to integrate.
- Break up difficult rational functions by following these two steps:
  1. Write the function as the sum of fractions with unknown numerators.
  2. Solve for the unknown numerators.

Using the technique of partial fractions

Consider $\int \frac{5}{(2x+1)(x-2)}dx$.

Notice that this integral can't be evaluated with a *u*-substitution. Consider using a **partial fraction decomposition** instead.

Step 1 First write the function as the sum of two fractions with unknown numerators.

When breaking a **rational expression** into **partial fractions**, start by factoring the denominator. These factors are used as the denominators of the partial fractions. Use unknowns for the numerators.

$$\frac{5}{(2x+1)(x-2)} = \frac{A}{(2x+1)} + \frac{B}{(x-2)}$$

Find a common denominator.

$$= \frac{A(x-2)}{(2x+1)(x-2)} + \frac{B(2x+1)}{(2x+1)(x-2)}$$

Now find a common denominator for the expression. Notice that the common denominator matches the original expression's denominator.

$$= \frac{Ax-2A+2Bx+B}{(2x+1)(x-2)}$$

Combine the terms together.

$$\boxed{\frac{5}{(2x+1)(x-2)} = \frac{(A+2B)x+(B-2A)}{(2x+1)(x-2)}}$$

Finally, gather terms with like variables and factor them.

Step 2 Now solve for the unknown values.

Now you are ready to solve for the constant unknowns. Notice that each piece of the numerator of the final expression matches up to a term in the numerator of the original rational expression. Set these pieces equal to each other and solve the resulting system of equations.

$A+2B=0$ $\quad$ $B-2A=5$

Subtract $2B$ from both sides.

$A=-2B$ $\quad$ $B-2(-2B)=5$

$B+4B=5$ Substitute the value of A for B.

$5B=5$ Combine terms.

So $A=-2$ and $B=1$.

Once you have solved for the unknown numerators, plug them back into the original partial fraction.

Deconstructing the fraction gives the following result:

$$\boxed{\frac{5}{(2x+1)(x-2)} = \frac{-2}{(2x+1)} + \frac{1}{(x-2)}}$$

It's an IDENTITY

The statement you have just found is an identity. Moreover, since the expressions are equal you can substitute one for the other in the original integral.

Now evaluate the integral.

$$\int \frac{5}{(2x+1)(x-2)}dx = \int\left(\frac{-2}{2x+1} + \frac{1}{x-2}\right)dx$$

Go back to the original integral and make that substitution.

$$= -2\int \frac{1}{2x+1}dx + \int \frac{1}{x-2}dx$$

Notice that the new integral can be solved with simple *u*-substitutions.

$$= \boxed{-\ln|2x+1| + \ln|x-2| + C}$$

Don't forget your constant of integration.

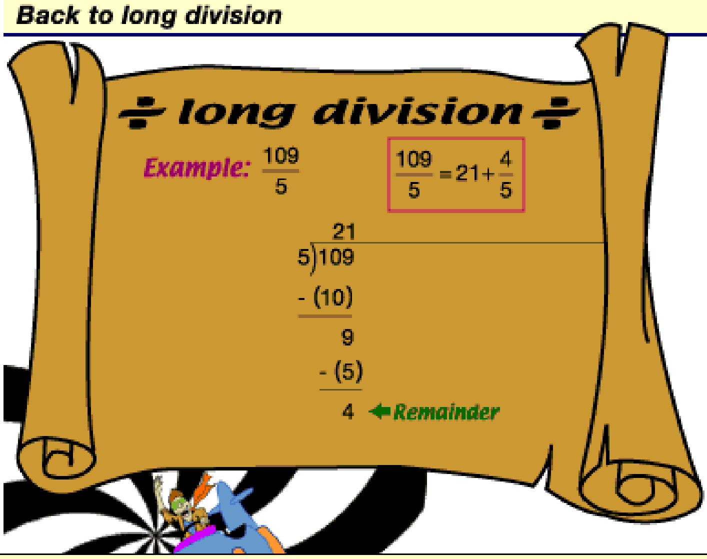
Unit: Techniques of Integration **Module:** Introduction to Integration by Partial Fractions

Long Division

key concepts:

- Use **polynomial long division** when the degree of the numerator of a rational function is higher than the degree of its denominator.
- The resulting expression can be integrated using the power rule.

Back to long division



Long division is an arithmetic technique for dividing a number when its numerator is greater than its denominator.

When using long division, the divisor (denominator) is applied to each digit of the dividend (numerator) systematically. What the divisor does not divide out is added into the next unit until no more units remain. The leftover value is called the remainder.

The remainder can be expressed as a fraction if you divided it by the denominator or divisor.

A similar technique from algebra can be used to simplify rational functions. This technique is called **polynomial long division**.

Polynomial long division

$$
\begin{array}{r}
x^2 + x + 2 \\
x-1\,\overline{)\,x^3 + 0x^2 + x} \\
-\left(x^3 - x^2\right) \\
\hline
x^2 + x \\
-\left(x^2 - x\right) \\
\hline
2x \\
-\left(2x - 2\right) \\
\hline
\text{Remainder} \Rightarrow 2
\end{array}
$$

So $\dfrac{x^3+x}{x-1} = x^2 + x + 2 + \dfrac{2}{x-1}$

In polynomial long division, the leading term of the divisor is compared to the leading term of the dividend. The expression necessary to make the divisor's leading term match the dividend's is written down and multiplied by the divisor. The result is subtracted from the dividend. The process repeats until the divisor is larger then the leftover expression.

Notice that just like in long division, the remainder is expressed as a fraction. Just place the remainder over the divisor.

Q: How is long division helpful?

A: It allows you to rewrite the integral in a form you can integrate by sight.

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2 + x + 2 + \frac{2}{x-1}\right)dx$$

Evaluate this by integrating term by term.

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

The reason long division is useful in calculus is because rational functions are not easily integrable. But polynomial functions are easy to integrate, as is the function $1/u$. Using long division lets you rewrite the complicated rational function as a sum of expressions easier to integrate.

Notice that all of these terms can be integrated with the power rule. Notice that a small u-substitution was used on the final term.

So algebra makes this problem solvable!