Calculus Lecture Notes

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Unit: Sequences and Series **Module:** Sequences

The Limit of a Sequence

key concepts:

- A **sequence** is a pattern of numbers.
- If a sequence approaches a particular number as the index *n* gets larger, then the sequence is said to **converge** to that number. If not, the sequence **diverges**.

Sequences of numbers

Example! The sequence of natural numbers

$$\{n\} = \{1,2,3,...\}$$

$\{a_n\} = \{1,2,3, ...\}$, where $a_n = n$.

Use the following notation:
a_n denotes the n-th term of the sequence $\{a\}$.
$\{a_1, a_2, a_3, ..., a_n, ...\}$ denotes a sequence.

A **sequence** is a collection of numbers. More importantly, a sequence is ordered. That means you can't rearrange the terms—doing so changes the sequence.

Sequences are notated in braces like this: $\{a_n\}$.

a is the name of the sequence. The subscript n tells you which term of the sequence you are talking about. If the subscript is just n, you're talking about a general term.

Divergent sequences

Example! The sequence of natural numbers

$$\{n\} = \{1,2,3,...\}$$

$\{a_n\} = \{1,2,3, ...\}$, where $a_n = n$.

Question What happens to the terms of the sequence $\{1,2,3, ...\}$ as n gets larger?

Answer As n gets larger, the numbers in the sequence approach infinity.

The sequence diverges!

The natural numbers is an example of a sequence. Notice that the subscript tells you which term of the sequence you are talking about. So $a_4 = 4$. The subscript is called the index.

One of the important properties of a sequence is what happens to the terms as the index gets larger. If the sequence approaches a number as the index increases, then the sequence has a limit.

In this particular sequence, the terms get bigger and bigger as the index increases. The terms of the sequence approach infinity and so the sequence does not have a limit.

A sequence that does not have a limit **diverges**.

Convergent sequences

Consider $\{a_n\}$ where $a_n = \frac{1}{n}$.

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ..., \frac{1}{n}, ...\right\}$$

Sequence of reciprocals

As the denominator gets larger the terms approach zero.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

Since the limit exists, the sequence converges.

Here is another sequence to consider. The terms of the sequence are defined by the equation:

$$a_n = \frac{1}{n}.$$

This is a sequence of reciprocals. Notice that by using an equation to define how the terms act you save yourself the trouble of writing them all out.

What is the limit of this sequence?

If you take the limit of the equation that describes the terms as n approaches infinity, you will see that the terms approach zero.

A sequence that has a limit is said to **converge**.

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Determining the Limit of a Sequence

key concepts:

- If a **sequence** approaches a particular number as the index n gets larger, then the sequence is said to **converge** to that number. If not, the sequence **diverges**.
- You can sometimes use **L'Hôpital's rule** to evaluate the limit of a sequence.

How to determine the limit of a sequence

Example! Find the limit of the sequence $\{n^2\}$, where n is any natural number.

$$\lim_{n \to \infty} n^2 = \infty$$

As n gets large, so does n^2.

The sequence diverges.

Remember that a **sequence** is an ordered collection of numbers. Consider this sequence that represents the squares of the natural numbers.

The limit of the sequence can be found by taking the limit of the a_n term as n approaches infinity. Since the limit does not exist, the sequence **diverges**.

How to determine the limit of a sequence

Let's try another!

Example! Find the limit of the sequence $\{\sin(\pi n)\}$.

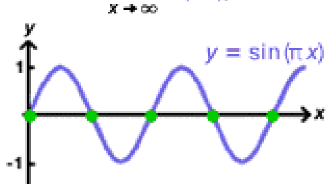
$$\{\sin(\pi n)\} = \{\sin(\pi), \sin(2\pi), \sin(3\pi), \ldots\}. = \{0,0,0,\ldots\}$$

$$\lim_{n \to \infty} \sin(\pi n) = \lim_{n \to \infty} 0 = 0$$

The sequence converges to zero.

Consider this . . .

$\lim_{x \to \infty} \sin(\pi x)$, where x is any real number.



Because sine oscillates between -1 and 1, the limit of $\sin(\pi x)$ does not exist,... whereas the limit of the associated sequence $\{\sin(\pi n)\}$ does exist.

Consider this trickier example.

Sometimes it is a good idea to write the sequence out so you can see patterns that might develop. Since the argument of the sine function is a multiple of π and since n can only be a natural number, then the sine function will only equal zero.

The limit of the sequence equals zero. Since the limit exists, the sequence **converges**.

Notice that although the sine function itself does not converge when you take the limit, the sequence does. The sequence has the advantage of only using the natural numbers whereas the function is defined by all the other real numbers as well.

Remember that the braces { } indicate a sequence.

L'Hôpital's rule for finding limits of sequences

Find the limit of the sequence $\left\{\dfrac{n^2 - n}{3n^2 + 1}\right\}$.

Notice that $\lim_{n \to \infty} \dfrac{n^2 - n}{3n^2 + 1}$ is an indeterminate form.

Use L'Hôpital's rule.

$$\lim_{n \to \infty} \frac{n^2 - n}{3n^2 + 1} = \lim_{n \to \infty} \frac{2n - 1}{6n}$$

$$= \lim_{n \to \infty} \frac{2}{6}$$

$$= \lim_{n \to \infty} \frac{1}{3}$$

$$= \boxed{\frac{1}{3}}$$

Use L'Hôpital's rule twice, because you still have infinity over infinity after using it once.

The limit of the sequence is $\frac{1}{3}$.

When looking for the limit of the sequence you can use all of the techniques you know for finding limits. The **limit laws** and **L'Hôpital's rule** all apply.

Consider the limit of this sequence. The limit produces an **indeterminate form**. So apply L'Hôpital's rule.

You can even reapply L'Hôpital's rule if you need to. All the rules you learned before still apply.

The sequence has a limit and therefore converges.

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The Squeeze and Absolute Value Theorems

key concepts:

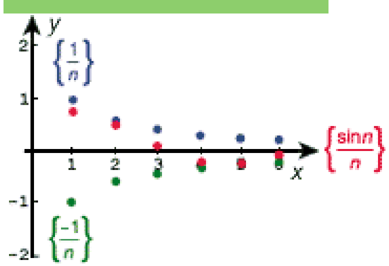
- Given three **sequences** $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, the **squeeze theorem** states that if $b_n \le a_n \le c_n$ for all $n \ge N$ where N is some natural number and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.

- The **absolute value theorem** states that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Squeezing a limit

Consider $\lim_{n\to\infty} \frac{\sin n}{n}$.

$\frac{\sin n}{n}$ is between two sequences whose limits are known.



SQUEEZE

For all values of n, $-1 \le \sin n \le 1$

Therefore $\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$$\lim_{n\to\infty} \frac{-1}{n} \le \lim_{n\to\infty} \frac{\sin n}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$0 \le \lim_{n\to\infty} \frac{\sin n}{n} \le 0$$

So $\boxed{\lim_{n\to\infty} \frac{\sin n}{n} = 0}$

You can evaluate the limit of a sequence by comparing that sequence to other sequences.

Here you can state that the value of sine is always between –1 and 1. Therefore you can create an identity relating the given sequence to two others.

Now compare the limits of these sequences. Notice that you can easily evaluate the limits on the far left and far right.

Since the sequences around the unknown sequence both converge to the same value, then they restrict the possible values of the unknown sequence. They squeeze the sequence down to only one possible answer. This theorem is called the **squeeze theorem**.

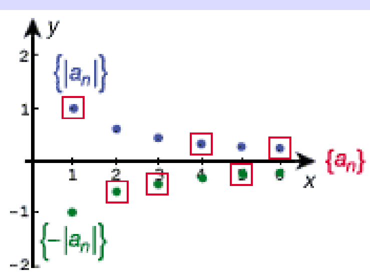
A result of the squeeze theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

This is a result of the squeeze theorem.

$$\lim_{n\to\infty} |a_n| = 0 \qquad \lim_{n\to\infty} -|a_n| = 0.$$

$$-|a_n| \le a_n \le |a_n|$$



The **absolute value theorem**: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Consider a sequence for which you can't determine the limit. If you can prove that the limit of the absolute value of the sequence approaches zero, then you can use the squeeze theorem to prove that the original sequence also approaches zero.

Notice that if you have a sequence, a_n, then the absolute value of the terms of that sequence will always be greater than or equal to a_n.

Likewise, if you multiply the absolute value by negative one the resulting sequence will always be less than or equal to a_n.

So you can sometimes use absolute values to find two sequences that squeeze a third sequence to prove that it converges

This theorem is called the **absolute value theorem**.

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