Unit: Sequences and Series **Module:** Taylor and Maclaurin Series

Taylor Series

key concepts:

- The **Taylor series** expansion of $f(x)$ centered at $x = c$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ assuming that $f(x)$ is differentiable an infinite number of times.
- For a given x, if the Taylor series expansion of $f(x)$ converges then it equals $f(x)$.

Review of Taylor polynomials

The k^{th} Taylor polynomial centered about $x = c$ of $f(x)$ is:

$$\sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The Taylor polynomial is a good approximation of $f(x)$ when x is near c and the associated error of the approximation is $R_n(x)$.

A **Taylor polynomial** has a finite degree k. As a result, it only approximates the values of the original function.

The approximation is best near the center, c, of the polynomial. The error of the approximation is the absolute value of the remainder.

Taylor series

What if you want an exact value of $f(x)$?

Since the approximation improves as k increases, let k go to infinity.

Taylor series of $f(x)$ centered at $x=c$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n,$$ given that $f(x)$ is differentiable an infinite number of times.

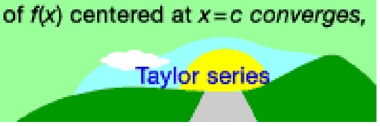
The Taylor series centered at $x=0$ is known as a Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$, where $f(x)$ is differentiable an infinite number of times.

IMPORTANT FACT!

For x such that the Taylor series of $f(x)$ centered at $x=c$ *converges*,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$



To get an exact value using a Taylor polynomial, you have to let k go to infinity. You can only do this if the function has infinitely many derivatives.

By replacing k with infinity, you can create a **Taylor series**.

The Taylor series centered at $x = 0$ is called a **Maclaurin series**.

Now you can use series tests to determine if and when a given Taylor series converges. If the Taylor series of a function $f(x)$ converges for a given value of x, then the Taylor series is actually equal to the function. In this case there is no error!

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Examples of the Taylor and Maclaurin Series

key concepts:

- The **Maclaurin series** expansion of $f(x)$ is the **Taylor series** expansion of $f(x)$ centered at $x = 0$.
- For a given x, if the Taylor series expansion of $f(x)$ converges then it equals $f(x)$.

The Taylor series expansion of e^x

Example: $f(x) = e^x$ around $c = 0$

1 Find the derivatives of $f(x)$: $f^n(x) = e^x$

2 Evaluate $f^n(x)$ at $x = 0$: $f^n(0) = e^0 = 1$

3 Plug the derivatives into the formula for the Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \qquad c = 0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(x)^n \qquad f^n(0) = e^0 = 1$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Because the infinite series converges for all x, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

The steps for determining a **Taylor series** are the same as those for finding a **Taylor polynomial**.

First find all the derivatives of the function. Since you want a series, the function has to have infinitely many derivatives. You must find all of them. In this case, all the derivatives of the exponential function are the same

Second, evaluate the derivatives at the center, c. Here the center is zero. All of the derivatives evaluated at zero are equal to one.

Third, plug the values into the Taylor series formula.

The Taylor series expansion of e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

This series converges by the **ratio test**, so it actually equals e^x.

The Taylor series expansion of sinx and cosx

Example: $f(x) = \sin x$ around $c = 0$

1 Find the derivatives of $f(x)$:

2 Evaluate $f^n(x)$ at $x = 0$:

$f^{(n)}(x)$	$x = 0$
$f^{(0)}(x) = \sin x$	$= 0$
$f^{(1)}(x) = \cos x$	$= 1$
$f^{(2)}(x) = -\sin x$	$= 0$
$f^{(3)}(x) = -\cos x$	$= -1$
$f^{(4)}(x) = \sin x$	$= 0$
$f^{(5)}(x) = \cos x$	$= 1$

Only the odd indexed terms contribute to the series.

The values of these derivatives alternate between 1 and –1.

3 Plug the derivatives into the formula for the Taylor series expansion of sinx:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

When $n = 0$, $2n + 1 = 1$, when $n = 1$, $2n + 1 = 3$ and so on, producing all the odd numbers.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n)}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Sine also has infinitely many derivatives, but they are not equal. However, they do follow a pattern. The fifth derivative is the same as the first. So after every four derivatives the pattern repeats.

When you evaluate the derivatives at zero, those involving sine are equal to zero. Only the odd indexed derivatives will contribute to the series. Those values alternate between 1 and –1.

When you plug the values into the Taylor series formula, you get $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$.

You have to do some fancy work with the index to produce the closed form.

The Taylor expansion for cosine can be found in the same manner. Both series converge for all x, so the expansions equal their respective functions.

Calculus Lecture Notes

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Unit: Sequences and Series **Module:** Taylor and Maclaurin Series

New Taylor Series

key concepts:

- A powerful technique for finding the **Taylor series** of a composite function is to use a change of variables on the Taylor series of the related elementary function.

New Taylor series

What is the **Maclaurin series** for e^{x^2}?

$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ when $c = 0$.

Find all derivatives and look for a pattern that you can substitute into the equation for the **Maclaurin series**.

Substitute x^2 for x in the **Maclaurin series** of e^x.

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

There are two ways to find the **Maclaurin series** (or **Taylor series** centered at $c = 0$), of a composite function.

One way is to calculate all the derivatives of the function and evaluate them at $x = 0$.

A second way is to substitute the second function of the composition into the Maclaurin series formula for the elementary function.

In this case, the exponential function is an elementary function whose Maclaurin series expansion you know.

Substituting x^2 for x produces the Maclaurin series expansion you wanted.

What is the **Maclaurin series** for e^{2x}?

$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ when $c = 0$.

Substitute $2x$ for x in the **Maclaurin series** of e^x.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

If you want, you can use algebra to make the result look more like the Maclaurin series formula.

New Taylor series

What is the **Maclaurin series** for $\cos(x^2)$?

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Substitute x^2 for x in the **Maclaurin series** for $\cos(x)$.

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

Notice how simple it is to find the Maclaurin series expansion for $\cos(x^2)$.

You just recall the expansion for $\cos x$ and then replace x by x^2.

If you used the definition of Maclaurin series, you would have to take infinitely many derivatives of $\cos(x^2)$. They get complicated, because the first derivative involves the chain rule and all the following derivatives require the product rule as well as the chain rule.

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Convergence of Taylor Series

key concepts:

- It is useful to know the values of x for which the **Taylor series** expansion of a function converges because for those values the expansion will equal the function.
- To find the interval over which a Taylor series converges, apply the ratio test. Use a different test to determine if the series converges at the endpoints of the interval.

The Taylor series of f(x) = lnx

example: Consider $f(x) = \ln x$ centered at $c = 1$.

Q: What is the Taylor series of $f(x) = \ln x$?

Plug the derivatives into the formula for the Taylor series:

$f(x) = \ln x \quad = 0$

$f^{(1)}(x) = \frac{0!}{x} \quad = 1$

$f^{(2)}(x) = \frac{-(1!)}{x^2} \quad = -1$

$f^{(3)}(x) = \frac{2!}{x^3} \quad = 2!$

$f^{(4)}(x) = \frac{-(3!)}{x^4} \quad = -(3!)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

$$= 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

a: $$\boxed{= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n}$$

Since this **Taylor series** is not centered at $c = 0$, it is not a **Maclaurin series**.

The natural logarithm function has infinitely many derivatives. Start by finding the first few and then look for a pattern. For this function the signs are alternating, the numerators are factorials, and the denominators are increasing powers of x.

Next you evaluate the derivatives at $c = 1$. Notice that the denominators disappear.

Finally, plug the values into the formula for the Taylor series. Watch how the factorials cancel.

Since the natural log of one is zero, the first term is zero. Just change the index so that n starts at one, instead of zero.

Intervals for which the Taylor series of lnx converges

Q: For what values of x does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ converge?

Use the ratio test!

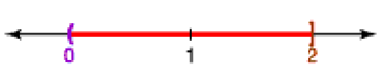
$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{n+1}(x-1)^{n+1}}{\frac{(-1)^{n+1}}{n}(x-1)^n} \right| = \lim_{n\to\infty} \frac{n|x-1|^{\cancel{n+1}}}{n+1\,\cancel{|x-1|^n}}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}|x-1|$$

$$= \boxed{|x-1|}$$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ will converge absolutely when $|x-1| < 1$.

Set $|x-1| < 1$, then $0 < x < 2$.



Now check the endpoints!

If $x = 0$, the Taylor Series of $\ln x$ diverges.
If $x = 2$, the Taylor Series of $\ln x$ converges.

If $x = 2$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Now that you have the Taylor series expansion, you want to know the values of x that make it converge.

Start with the **ratio test**. Remember that n is the only variable that matters to the limit. For now x is just a constant.

The absolute value symbols eliminate the powers of -1. Just keep them around the $(x - 1)$ factors.

The ratio test says that the series converges if the limit is less than one, but it is inconclusive if the limit equals one.

Think about absolute value as indicating distance. The points $x = 0$ and $x = 2$ are both one unit away from $x = 1$.

Don't forget to check the endpoints! If $x = 0$, the series is the negative **harmonic series**, which diverges. That make sense, because the natural log function is not defined for $x = 0$.

For $x = 2$, the series is the **alternating harmonic series**, which converges.