Unit: Sequences and Series **Module:** The Alternating Series

Alternating Series

key concepts:

- An **alternating series** is a series whose terms alternate between positive and negative values.
- The **alternating series test** states that an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if the a_n values are positive, decreasing and approaching zero.

Alternating series are series with terms that alternate between positive and negative values.

Example! Consider the following series where $a_n > 0$.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

Alternating series get their name from the fact that the terms alternate between being positive and negative.

Raising (–1) to the n+1th power makes it alternate between –1 and +1. Make sure that a_n is positive, though.

Q. How do you know if an alternating series converges?

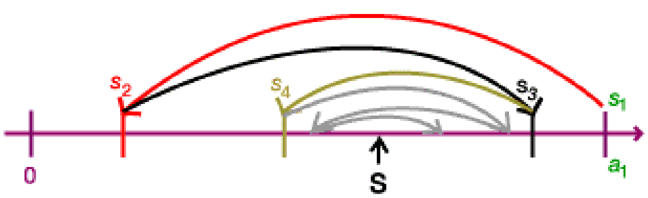
Think about the sequence of partial sums.

$s_1 = a_1$

$s_2 = a_1 - a_2$

$s_3 = a_1 - a_2 + a_3$

$s_4 = a_1 - a_2 + a_3 - a_4$



For any kind of series, the important information to know is whether it converges or diverges.

Suppose each a_n is smaller than the one before and the signs are alternating. That means that either you are subtracting less than the current sum or you are adding less than was just subtracted.

Alternating between adding and subtracting means that the partial sums begin to approach some value, S. Thus the series converges if a_n is approaching zero. This is the basis for the **alternating series test**.

The alternating harmonic series

Example! $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

Q. Does the series converge?

Using the alternating series test, look at $\left\{\frac{1}{n}\right\}$.

1. The terms are all positive, $\frac{1}{n} > 0$.
2. The terms are decreasing, $\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \cdots$.
3. The limit is 0, $\lim_{n\to\infty} \frac{1}{n} = 0$.

a. Yes, the series converges!

This series, called the alternating harmonic series, consists of the terms of the harmonic series with alternating signs.

There are three steps you have to take when using the alternating series test.

- Check the sign of the non-alternating part of the terms.
- Verify that the non-alternating part is decreasing.
- Make sure the non-alternating parts are approaching zero.

Since all the conditions are met, the series converges by the alternating series test.

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The Alternating Series Test

key concepts:

- The **alternating series test** states that an **alternating series** $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if the a_n values are positive, decreasing and approaching zero.
- Be sure that you apply the alternating series test only to alternating series.

Using the alternating series test

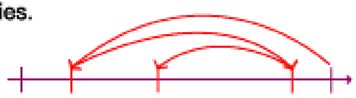
Example! $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3+1}$

Any time you see $(-1)^n$ or $(-1)^{n+1}$, you know the series is alternating and you can use the alternating series test.

An alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

1. $a_n > 0$
2. $a_1 > a_2 > a_3 > \cdots$
3. $\lim_{n\to\infty} a_n = 0$

Important! If the limit of the terms is zero in a non-alternating series, you cannot conclude that the series converges. Don't use the alternating series test on non-alternating series.



An **alternating series** alternates because it has a factor of –1 raised to a power that fluctuates between even and odd values. This is done by raising –1 to the n power or the $n+1$ power.

An alternating series must meet three conditions before you can say it converges by the **alternating series test**.

- The non-alternating part (a_n) must be positive.
- The a_n values must be decreasing.
- The limit of the a_n values must be zero.

Remember how the alternating series test works. Each step reverses most of what happened in the previous step.

If a series does not alternate, its terms may approach zero, but not converge. Do not use the alternating series test on non-alternating series.

Don't forget the "quicky test"

Example! $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{3n-1}$

Q. Does the series converge?

Look at $\frac{n}{3n-1}$.

✓ 1. The terms are all positive, $\frac{n}{3n-1} > 0$

✓ 2. The terms are decreasing, $\frac{1}{3(1)-1} > \frac{2}{3(2)-1} > \frac{3}{3(3)-1} > \frac{4}{3(4)-1}$

$$\frac{1}{2} > \frac{2}{5} > \frac{3}{8} > \frac{4}{11}\cdots$$

$$0.5 > 0.4 > 0.375 > 0.364\cdots$$

3. $\lim_{n\to\infty}\frac{n}{3n-1} = \frac{1}{3} \neq 0$!!! Use L'Hôpital's rule.

This series is an alternating series because it has a factor of –1 raised to the $n+1$ power.

The a_n values are positive and decreasing.

The limit of the a_n values is not zero. If you first use the **nth term test** (or "quicky test"), you can save yourself some work.

This series diverges by the quicky test.

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Estimating the Sum of an Alternating Series

key concepts:

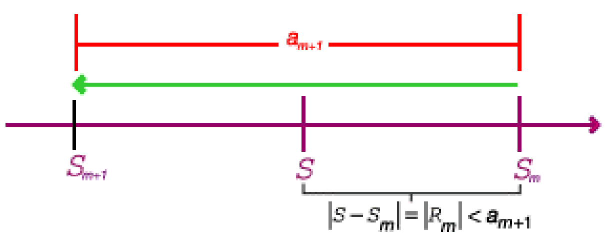
- You can approximate the sum of an infinite series by adding up its first *m* terms.
- The error introduced when you approximate the sum *S* of an **alternating series** $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ by the partial sum S_m of its first *m* terms is a_{m+1}.

Estimating the sum of an alternating series

To estimate the sum of an infinite series you could add up the first *m* terms of the series and ignore the rest.

Example! $S=\sum_{n=1}^{\infty}(-1)^{n+1}a_n$

Consider the mth partial sum, $S_m=\sum_{n=1}^{m}(-1)^{n+1}a_n$.



What is the error in S_{m+1} as an estimate of S?

Note that $|S-S_m|=|R_m|<a_{m+1}$ and therefore a_{m+1} is an upper limit for the error $|R_m|$ in this approximation of S.

The *m*th partial sum S_m of a series *S* acts as an approximation for the sum of the series.

For an alternating series *S*, each a_n value overshoots the sum of the series. Therefore the distance between *S* and S_m must be less than the next term, a_{m+1}.

Remember that the distance between two values is represented by the absolute value of the difference.

R_m is the remainder when you subtract S_m from *S*.

Now you can determine how close your approximation is to the true value. It's accurate to within a_{m+1}.

The error of approximations

Example! $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=S$

Q: Can you approximate the series to three decimal places?

You want $|S-S_m|<0.0001=\frac{1}{100^2}=a_{100}$ $\quad a_n=\frac{1}{n^2}$

You found that $a_{100}=a_{m+1}$, so *m* is 99

The 99th partial sum will give you an answer that is within 0.0001 of the actual value of the series.

a: $S_{99}=\sum_{n=1}^{99}\frac{(-1)^{n+1}}{n^2}=\boxed{0.822}517\cdots$

You can be sure that the first three digits of this answer are correct.

$|S-S_m|=|R_m|<a_{m+1}$

You can also determine which partial sum will produce an approximation with a given accuracy.

In this example you want the approximation to be true for three decimal places. In other words you want the error to be less than 0.0001.

For this series, 0.0001 equals a_{100}, which matches up with a_{m+1}. Therefore *m* is 99.

Using a computer or programmable calculator, you can determine the 99th partial sum.