Unit: Parametric Equations and Polar Coordinates **Module:** Understanding Polar Coordinates

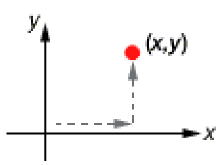
The Polar Coordinate System

key concepts:

- In **polar coordinates**, points are identified by their distance r from the origin (or **pole**) and their angle θ from the horizontal right axis.
- Both r and θ are directed, meaning they can be either positive or negative. The angle θ is positive in the counterclockwise direction and negative in the clockwise direction. The distance r is positive in the direction of the given angle and negative in the opposite direction.

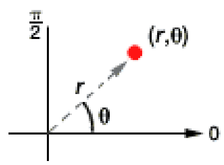
Comparison between polar and Cartesian coordinates

Cartesian Coordinates



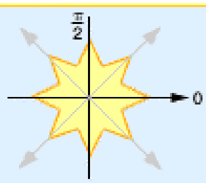
The x-coordinate (horizontal distance from the origin) and the y-coordinate (vertical distance from origin) describe the point's position.

Polar Form



The distance, r, from the origin and the angle θ from the horizontal axis describe the point's position.

Think of the polar coordinate system as rays of length r and angle θ that emanate out from the pole.



When describing a point on the Cartesian plane, you use an ordered pair of x-coordinates and y-coordinates. The x-coordinate tells you the horizontal distance and the y-coordinate tells you the vertical distance.

But there are other ways to describe the location of points. One such way is to use the **polar coordinate system**.

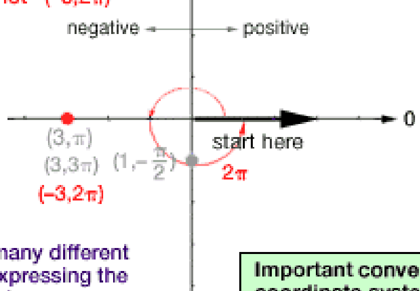
The polar coordinate system uses an angle and one distance instead of the horizontal and vertical distances. The angle tells you how far to move **counterclockwise** from the positive horizontal axis and the distance tells you how far out to travel in that direction.

The origin is often called the **pole** in **polar coordinates**.

Plotting polar coordinates

Example!

Plot $(3,\pi)$
Plot $(3,3\pi)$
Plot $(1,-\frac{\pi}{2})$
Plot $(-3,2\pi)$



There are many different ways of expressing the same point.

The point $(-3,2\pi)$ is the same point as $(3,\pi)$ and $(3,3\pi)$.

Important conventions in the polar coordinate system

1. Measure θ from the right side of the horizontal axis.
2. If the angle θ is positive, move counterclockwise.
3. If the angle θ is negative, move clockwise.

Suppose you wanted to find the point $(3,\pi)$. The first coordinate tells you the distance, r. The second coordinate tells you the angle, θ. So you must first move counterclockwise θ radians and then move out 3 units.

Notice that there are several ways to get to the same point in the polar coordinate system. The point $(3,3\pi)$ ends up at the exact same place as the point $(3,\pi)$.

Negative angles indicate clockwise movement. Negative distances indicate moving in the opposite direction of the angle. So the point $(-3,2\pi)$ is another point that lands you on top of $(3,\pi)$.

Another interesting fact about polar coordinates is that if you are given a constant angle but a variable distance you generate a line. If you are given a constant distance but a variable angle you generate a circle.

Unit: Parametric Equations and Polar Coordinates **Module:** Understanding Polar Coordinates

Converting Between Polar and Cartesian Forms

key concepts:

- To convert from Cartesian to **polar coordinates**, use the formulas:

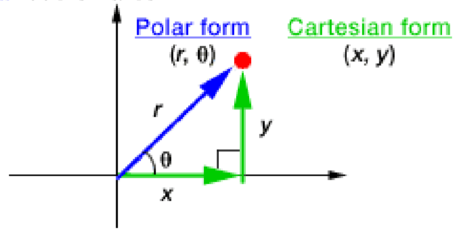
$$r = \sqrt{x^2 + y^2}\,, \qquad \theta = \arctan\frac{y}{x}\,.$$

- To convert from polar to Cartesian coordinates, use the formulas:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Converting between polar and Cartesian coordinates

Q: How do you convert back and forth between Cartesian and polar coordinates?



A: To convert from polar to Cartesian coordinates:

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

To convert from Cartesian to polar coordinates:

$$x^2 + y^2 = r^2 \qquad \theta = \arctan\frac{y}{x}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arcsin\frac{y}{\sqrt{x^2+y^2}}$$

$$\theta = \arccos\frac{x}{\sqrt{x^2+y^2}}$$

It is often useful to move back and forth between the Cartesian and **polar coordinate** systems. You can do this because both systems describe the position of the same object.

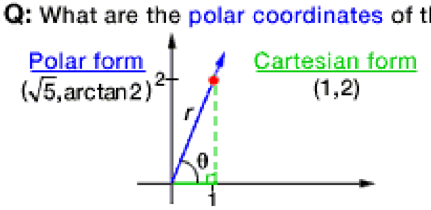
To convert from polar to Cartesian coordinates, you need to express the x-distance and y-distance in terms of r and θ. Recall from trigonometry that cosine is determined by the x-distance divided by the hypotenuse of the right triangle. If you rearrange the definition you get an expression for x. A similar method works for y and sine.

To convert from Cartesian to polar coordinates, you need to express x and y in terms of the distance and angle. Since r is the hypotenuse of the triangle, it follows directly from the Pythagorean theorem. The angle can be found by using one of the inverse trigonometric functions.

Converting from Cartesian to polar forms

Example! $(x, y) = (1, 2)$

Q: What are the polar coordinates of this point?



To convert back and forth, just plug the values you have into the appropriate equation and simplify.

Here you have the Cartesian coordinates (1,2).

A: Use the Pythagorean theorem to find r:

$$x^2 + y^2 = r^2$$
$$1^2 + 2^2 = r^2$$
$$1 + 4 = r^2$$
$$5 = r^2$$
$$\boxed{r = \sqrt{5}}$$

Use trigonometric relationships to find θ:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{2}{1}$$
$$\tan\theta = 2$$
$$\boxed{\theta = \arctan 2}$$
$$\boxed{\theta = 1.107 \text{ radians}}$$

Since $x = 1$ and $y = 2$, plug those values into the Pythagorean theorem to get r.

You can use any of the inverse trigonometric functions to find θ since you have all three sides of the right triangle. Usually tangent will be the easiest however, since it uses the coordinates that you were given.

Find the value of tangent and then take the inverse tangent of each side. This isolates the θ-term.

Calculus Lecture Notes

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Unit: Parametric Equations and Polar Coordinates **Module:** Understanding Polar Coordinates

Spirals and Circles

key concepts:

- In **polar coordinates**, points are identified by their distance r from the origin (or **pole**) and their angle θ from the horizontal right axis.

Graphing a spiral in polar coordinates

Example $r = \theta$

In this graph, the change in radius corresponds to the exact same change in angle.

Q: What does the graph of this equation look like?

A: Plot a few points:

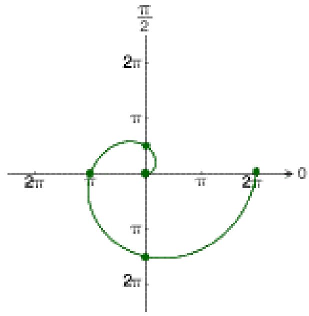
$\theta = 0,\ r = 0$

$\theta = \frac{\pi}{2},\ r = \frac{\pi}{2}$

$\theta = \pi,\ r = \pi$

$\theta = \frac{3\pi}{2},\ r = \frac{3\pi}{2}$

$\theta = 2\pi,\ r = 2\pi$



Q: Is $r = \theta$ a function?

A: No, because there is not a unique value of r for every θ. It fails the emanating ray test.

You have already seen that the graph of $r = k$ where k is a constant produces a circle centered at the pole and the graph of $\theta = k$ produces a line passing through the pole.

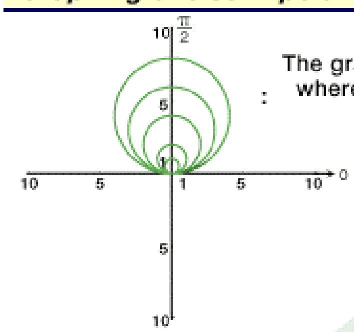
What happens when you set $r = \theta$?

As θ increases, so does r. So as you move around the **pole** you also move further away.

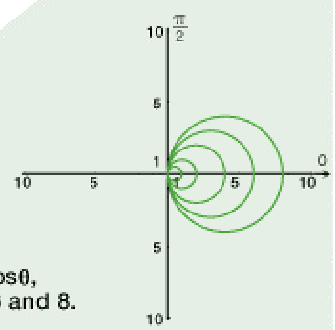
The end result is a spiral, also called the spiral of Archimedes.

Notice that this relationship is not a function, since you can have several values of r for the same value of θ. It fails the emanating ray test, since a ray starting at the pole hits the graph in more than one place.

Graphing circles in polar coordinates



The graph of $r = a\sin\theta$, where $a = 1, 2, 4, 6$ and 8.



The graph of $r = a\cos\theta$, where $a = 1, 2, 4, 6$ and 8.

The sine and cosine functions have very interesting behaviors in **polar coordinates**.

Start by considering sine. Notice that at first, the sine function is zero. Then it slowly increases until it reaches a maximum value at $\pi/2$. Then the graph moves back to zero at π, to a negative value at $3\pi/2$, and then back to its starting value at 2π.

It turns out that the sine function moves in a perfect circle in polar coordinates! You can even prove this fact if you want by using the Pythagorean theorem and the definitions of sine and cosine.

The cosine function behaves the same, but it starts at its maximum value. So the sine curve looks like a circle centered half-way from its maximum value on the positive vertical axis while the cosine curve looks like a circle centered half-way from its maximum value on the positive horizontal axis.

Graphing Some Special Polar Functions

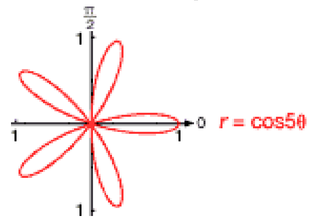
key concepts:

- In **polar coordinates**, points are identified by their distance r from the origin (or **pole**) and their angle θ from the horizontal right axis.
- Some of the important polar functions are the **rose curve**, the **cardioid**, the **limaçon**, and the **lemniscate**.

Rose curves

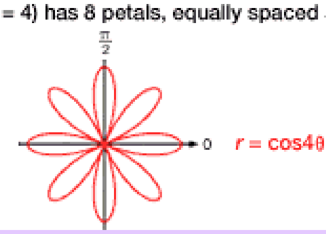
Rose curve $r = \cos n\theta$ n = constant

The graph of $r = \cos 5\theta$ ($n = 5$) has 5 petals, equally spaced around the pole.



In general, if n is an odd number, the graph of $r = \cos n\theta$ will have n equally-spaced petals, and one will always be centered on the ray $\theta = 0$.

The graph of $r = \cos 4\theta$, ($n = 4$) has 8 petals, equally spaced around the pole.



In general, if n is an even number, the graph of $r = \cos n\theta$ will have $2n$ equally-spaced petals.

The graphs of the sine and cosine functions resemble unit circles that touch the **pole**. By changing different constants in the function you can create different graphs.

The first such graph occurs when you multiply the argument of the sine or cosine function by a constant. These curves are called **rose curves**.

By multiplying the argument by a constant, you change the period of the trig function. Doing so increases the number of loops in your graph. The number of loops of the curve depends on the constant you used.

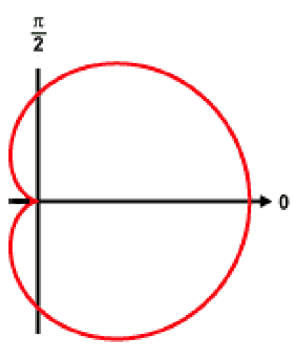
An odd constant produces a number of loops equal to the constant. An even constant produces a number of loops equal to twice the constant.

For cosine, the first loop is bisected by the positive horizontal axis. For sine, the first loop begins just above the axis.

The cardioid

Cardioid $r = a + b\cos\theta$ $a = b$ a and b are constants

The graph looks like a circle centered at $\theta = 0$ with a "dimple" at $\theta = \pi$.



Another way you can change the graphs of the sine and cosine functions is by adding a constant to the trig expression and multiplying the trig expression by another constant with the same value as the first.

If you think about this function you will notice that it is never negative. It equals zero when the trig function equals –1.

These functions create graphs known as **cardioids**, since they resemble hearts.

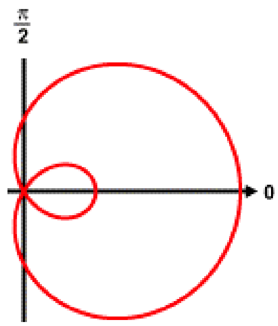
A cardiod generated with cosine looks like a heart on its side. A cardiod generated with sine looks like an upside down heart.

Graphing Some Special Polar Functions

The limaçon

Limaçon $r = a + b\cos\theta$ $a < b$ a and b are constants

The graph looks like a **cardioid** centered at $\theta = 0$ with a "loop" at $\theta = \pi$.



In a cardiod, the two constants *a* and *b* are equal to each other. If you set *a* less than *b*, then the expression can sometimes be negative.

When this happens, the graph changes to include an inner loop. That inner loop represents the points where the expression is negative.

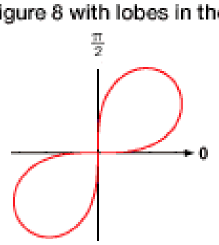
These curves are called **limaçons**.

If *b* is less than *a*, then the function never equals zero. When this happens, you get a limaçon with a dimple instead of an inner loop.

Lemniscate

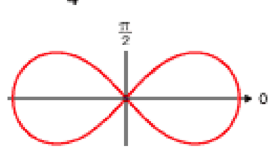
Lemniscate $r^2 = a^2\sin 2\theta$ a = constant

The graph looks like a figure 8 with lobes in the 1st and 3rd quadrants.



Another way of expressing a **lemniscate** is $r^2 = a^2\cos 2\theta$.

The graph looks like a figure 8 with lobes centered on $\theta = 0$. It is tangent to $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$



The last important polar graph is the **lemniscate**.

Lemniscates occur when the radius is squared and the argument of the trig function is multiplied by 2.

The lemniscate for sine looks like a figure-eight tilted so that it lies strictly in the first and third quadrant.

The lemniscate for cosine looks like an infinity symbol. It enters all four quadrants.

By changing the constants around, you can generate many more variations on these important polar coordinates.