Unit: Sequences and Series **Module:** Taylor and Maclaurin Polynomials

Taylor Polynomials

key concepts:

- Higher-degree **polynomial approximations** result in more accurate representations.
- To find a **Taylor polynomial**, find all the necessary derivatives and evaluate them at $x = c$, then insert the values into the Taylor polynomial function.

Defining Taylor polynomials

The k^{th} Taylor polynomial centered about $x = c$ of $f(x)$ is

mathematical $0! = 1$ convention

$$\sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!}(x-c)^n = f^{(0)}(c) + f^{(1)}(c)(x-c) + \frac{f^{(2)}(c)}{2}(x-c)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Higher-degree **polynomial approximations** can be represented using sigma notation (Σ). There is a mathematical convention that zero factorial equals one.

These polynomial approximations are called **Taylor polynomials**.

Using Taylor polynomials

Q: What is the fourth Taylor polynomial for $f(x) = \ln x$ centered at $c = 1$?

1 Find. 2 Evaluate. 3 Plug.

$f(x) = \ln x$

	$x=1$
$f^{(0)}(x) = \ln x$	$= 0$
$f^{(1)}(x) = \frac{1}{x}$	$= 1$
$f^{(2)}(x) = \frac{-1}{x^2}$	$= -1$
$f^{(3)}(x) = \frac{2}{x^3}$	$= 2$
$f^{(4)}(x) = \frac{-6}{x^4}$	$= -6$

A: $y = 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4$

linear

quadratic

cubic

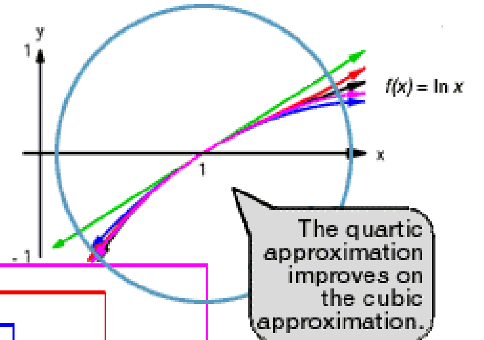
quartic

When deriving the Taylor polynomial for a given function, the first step is to **find** the derivatives of the function.

The second step is to **evaluate** the derivatives at the point about which the approximation is centered.

The third step is to **plug** the results into the formula for the Taylor polynomial.

When you derive the fourth Taylor polynomial, you get the lower polynomials for free.



quartic

cubic

quadratic

linear

A: $y = (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4$

When you look at the graphs of the approximations you can tell that each one does a better job than the previous.

The linear approximation was only good for values very close to the center $x = 1$.

The quartic approximation improves on all the previous approximations, as it hugs the curve of the natural log function much better.

Example: Find the value of $\ln(1.2)$

Use the quartic Taylor polynomial, which is an excellent approximation of ln at $x = 1.2$.

$$\ln(1.2) \approx (1.2-1) - \frac{(1.2-1)^2}{2} + \frac{2(1.2-1)^3}{6} - \frac{6(1.2-1)^4}{24}$$

$$= (0.2) - \frac{(0.2)^2}{2} + \frac{2(0.2)^3}{6} - \frac{6(0.2)^4}{24} \approx \boxed{0.182266\cdots}$$

Although you know the natural log of one is zero, you might not know the natural log of 1.2.

By plugging in 1.2, you can use your Taylor polynomial to approximate the value of $\ln(1.2)$.

The result agrees with the actual value of 0.18232155… to three significant figures.

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Maclaurin Polynomials

key concepts:

- The **Maclaurin polynomial** of $f(x)$ is the **Taylor polynomial** of $f(x)$ centered at $x=0$: $\sum_{n=0}^{k}\frac{f^{n}0}{n!}x^{n}=f^{(0)}(0)+f^{(1)}(0)x+\frac{f^{(2)}(0)}{2}x^{2}+\ldots+\frac{f^{(k)}(0)}{k!}x^{k}$.

Q: What is the Taylor polynomial when $c=0$?

A: $\sum_{n=0}^{k}\frac{f^{(n)}(0)}{n!}x^{n}=f^{(0)}(0)+f^{(1)}(0)x+\frac{f^{(2)}(0)}{2}x^{2}+\cdots+\frac{f^{(k)}(0)}{k!}x^{k}$

The k^{th} **Maclaurin polynomial** of $f(x)$ is the same as the Taylor polynomial of $f(x)$ centered about $x=0$.

If you center the **Taylor polynomial** approximation of a function about the point $x=0$, you get the **Maclaurin polynomial** approximation of the function.

Example: Consider $f(x)=e^{x}$

Q: What is the fourth degree Maclaurin polynomial of $f(x)=e^{x}$?

1. Find
2. Evaluate
3. Plug the derivatives into the formula for the fourth order Maclaurin polynomial.

Find	Evaluate at $x=0$
$f(x)=e^{x}$	
$f^{(0)}(x)=e^{x}$	$=1$
$f^{(1)}(x)=e^{x}$	$=1$
$f^{(2)}(x)=e^{x}$	$=1$
$f^{(3)}(x)=e^{x}$	$=1$
$f^{(4)}(x)=e^{x}$	$=1$

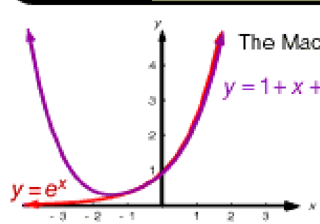
A: $y=1+x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}$

To find the fourth degree Maclaurin approximation of the exponential function e^{x}, you start by taking the first four derivatives. They are all equal to e^{x}.

Next you evaluate the function and all the derivatives at $x=0$. Since they are all the same, the results are all the same, namely one.

Substituting the values into the formula for the Maclaurin polynomial produces this expression.

Example: Find the value of $e^{0.1}$



$y=1+x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}$

$e^{0.1}\approx 1+0.1+\frac{1}{2}(0.1)^{2}+\frac{1}{6}(0.1)^{3}+\frac{1}{24}(0.1)^{4}$

$\approx 1+0.1+0.005+0.0001666+0.0000041$

$\approx 1.1051708\cdots$

$e^{0.1}=1.1051709\cdots$

Compare the graphs of the exponential function and your Maclaurin approximation. The Maclaurin approximation hugs the graph of the function very closely.

Now you can evaluate the exponential function at values like $x=0.1$.

The result agrees with the actual value to six decimal places.

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The Remainder of a Taylor Polynomial

key concepts:

- The **remainder** of an nth degree **Taylor polynomial** is $R_n(x) = \dfrac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$

  where z is between x and c.

- If x and c are close together, it is possible to accurately estimate the error term.

Review of Taylor polynomials

Taylor polynomials provide excellent approximations of complicated functions.

The n^{th} Taylor polynomial of $f(x)$ centered about $x = c$ is

$$P_n(x) = f(c) + f^{(1)}(c)(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \cdots \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$f(x) \approx P_n(x)$ near $x = c$.

Q: What is $P_n(x)$ at $x = c$?

A: $P_n(c) = f(c) + f^{(1)}(c)(c-c) + \frac{f^{(2)}(c)}{2!}(c-c)^2 + \cdots \frac{f^{(n)}(c)}{n!}(c-c)^n$

$P_n(c) = f(c) + f^{(1)}(c)(0) + \frac{f^{(2)}(c)}{2!}(0)^2 + \cdots \frac{f^{(n)}(c)}{n!}(0)^n$

$P_n(c) = f(c)$

At $x = c$, the Taylor approximation and the function itself are the same.

Taylor polynomial approximations are just that—approximations.

The approximation works best for values close to the center, c. Even for these values, the approximation is not usually equal to the original function.

Based on the definition of Taylor polynomials, the approximation is equal to the function at $x = c$.

The remainder of a Taylor approximation

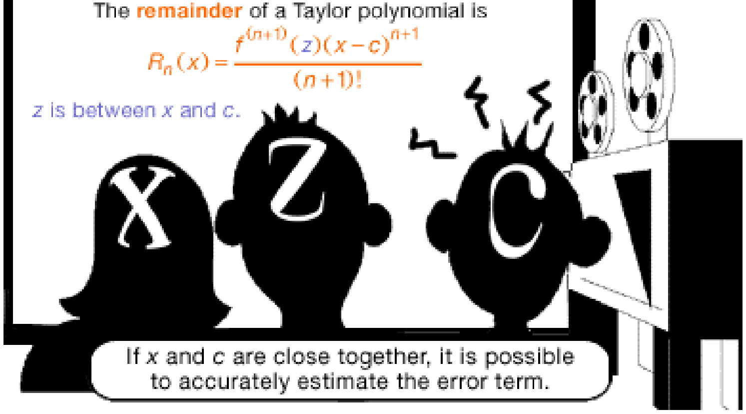
Near $x = c$, there is an error in the Taylor approximation called the **remainder**, $R_n(x)$ so that $f(x) = P_n(x) + R_n(x)$.

Because higher degree approximations are more accurate, you might guess that the error in $P_n(x)$ looks like the $(n+1)^{\text{th}}$ term.

The **remainder** of a Taylor polynomial is

$$R_n(x) = \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$$

z is between x and c.



If x and c are close together, it is possible to accurately estimate the error term.

Since the Taylor approximation is not exactly equal to the original function, there is an error term, called the **remainder**. If you add the remainder to the Taylor approximation, then the result *does* equal the original function.

For an nth degree Taylor approximation, the error term will look like the $(n+1)$th term.

Notice the value z in the remainder. All you know about z is that it lies somewhere between your value x and the center, c.

So if you are trying to approximate a function at an x-value close to the center (c) of the Taylor polynomial, then z will have less room to vary. You will be able to estimate the error more accurately.

Calculus Lecture Notes

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Unit: Sequences and Series **Module:** Taylor and Maclaurin Polynomials

Approximating the Value of a Function

key concepts:

- Since the z-value in the **remainder** of an nth degree **Taylor polynomial** lies between x and c, use the larger of $\dfrac{f^{(n+1)}(x)(x-c)^{n+1}}{(n+1)!}$ or $\dfrac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}$ as an upper limit.

Review of the error of Taylor polynomials

The remainder or error of a Taylor polynomial is

$$R_n(x)=\frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$$

for z between x and c

Example: The fourth degree Maclaurin polynomial of $f(x)=e^x$

$$y=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4$$

Recall that for Maclaurin polynomials $c=0$ and $f^{(n)}(0)=e^0=1$.

Estimate $e^{0.1}$

$$e^{0.1}\approx 1+0.1+\frac{1}{2}(0.1)^2+\frac{1}{6}(0.1)^3+\frac{1}{24}(0.1)^4$$

$$\approx 1+0.1+0.005+0.0001666+0.0000041$$

$$\approx 1.1051708\cdots$$

The **remainder** term of an nth degree **Taylor polynomial** is based upon the $(n+1)$th term.

The remainder expression contains a z-value that must lie between x and c. This may seem like a vague definition of z, but it is part of an approximation, after all. If you knew the exact value of z, that would mean you knew the exact value of the function, and you would not need an approximation.

Here is the value of $e^{0.1}$ that you obtained using the fourth degree **Maclaurin polynomial** approximation of the exponential function. Since Maclaurin polynomials are special types of Taylor polynomials, you can find their error, too.

Finding the error of a Taylor polynomial

Q: Can you estimate the error without using a computer?

A: $R_4(x)=\dfrac{e^z x^5}{5!}$ $\quad f^{(5)}(z)=e^z$

for z between 0 and x

The largest value that e^z could have is $e^{0.1}$ so

$$|R_4(0.1)|=\left|\frac{e^z(0.1)^5}{5!}\right|<\frac{e^{0.1}(0.1)^5}{5!}=9.209\text{x}10^{-8}$$

Approximate the error.

$0<z<0.1$

The fourth degree Taylor approximation has a very small error.

You can use the remainder expression to determine what the largest possible error is for your approximation.

When you use the values $n=4$ and $c=0$, you arrive at this expression for the remainder.

Since z is a value between c and x, it must lie between 0 and 0.1 in this case. Use the larger value when determining an upper limit for the remainder.

Although the remainder may be positive or negative, use the absolute value symbol to make the error positive.

Since 0.1 is close to the c-value of zero, this approximation is excellent.