Calculus Lecture Notes

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Unit: Sequences and Series **Module:** Power Series Representations of Functions

Differentiation and Integration of Power Series

key concepts:

- You can do calculus on a power series inside its interval of convergence.
- The derivative of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is $\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$.
- The integral of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is $\sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C$.

Differentiating and integrating power series **Power Series** generic brand $f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n$ Consider doing calculus on power series. For x-values within the interval of absolute convergence, you can differentiate and integrate $f(x)$.	Inside its interval of convergence a power series defines a function that you can differentiate and integrate.
Differentiating power series $f'(x)=\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$ The first term of the series is a constant and it becomes zero after differentiation. Technically this differentiation requires the chain rule, but the derivative of $(x-c)$ is one.	Since constant terms disappear during differentiation, you can start the index of the derivative at $n=1$.
Integrating power series For simplicity, consider the power series centered at zero, $f(x)=\sum_{n=0}^{\infty} a_n x^n$ $\int_0^x f(t)dt = \int_0^x a_0 t^0 dt + \int_0^x a_1 t^1 dt + \int_0^x a_2 t^2 dt + \cdots$ $= a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots$ $= \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ The general solution is: $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C$	There are many ways to integrate a power series. As long as you are in the interval of convergence they will all be equal. You can integrate a generic power series or the power series centered at $x=0$.
Differentiating the power series of e^x **example:** $e^x=\sum_{n=0}^{\infty} \frac{x^n}{n!}$ The radius of convergence for this power series is infinite. **Expect that:** $\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)=\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\frac{d}{dx}e^x = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$ $= \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!}$ $= \sum_{n=1}^{\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n}(n-1)!}$ $= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ $= \frac{x^{1-1}}{(1-1)!} + \frac{x^{2-1}}{(2-1)!} + \frac{x^{3-1}}{(3-1)!} + \cdots$ $= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$ $= \sum_{n=0}^{\infty} \frac{x^n}{(n)!} = \boxed{e^x}$	You can use the power series of e^x to prove that its derivative is e^x. It is convergent for all x-values. At first the derivative does not look much like the original power series. You can cancel the factor of n from the numerator by factoring the factorial in the denominator. Write out a few terms to see what pattern the derivative produces. It looks like the power series for e^x! Here is more proof that the derivative of e^x is e^x.

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Unit: Sequences and Series **Module:** Power Series Representations of Functions

Finding Power Series Representations by Differentiation

key concepts:

- You can do calculus on a power series inside its interval of convergence.
- The derivative of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is $\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$.
- You can find the power series of new functions by applying calculus to known power series.

Differentiating power series

Using calculus on power series is a powerful technique for finding alternative representations of functions.

example: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$ Differentiate the power series.

This function can be represented by a geometric power series. Notice that it only converges on the interval (–1, 1).

$$\frac{d}{dx}\left[(x-1)^{-1}\right] = (-1)(1-x)^{-2}(-1)$$ Use the chain rule.

$$= \frac{1}{(1-x)^2}$$

You can use the chain rule to differentiate the original function.

$$\frac{d}{dx}\left[(x-1)^{-1}\right] = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$ On the interval $|x|<1$ differentiate the power series term by term.

The derivative of the original function must be equal to the derivative of the power series for all x in the interval of convergence.

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n, \text{ for } |x| < 1$$

Use a change of variable so that the index will start at $n = 0$. That adds one more n-value, so you have to replace n with $n + 1$ inside the power series expression.

You can find power series for new functions by applying calculus to known power series!

Using derivatives you can now discover the power series expressions of new functions.

That's exciting by itself, but wait until you see what happens when you integrate!

Unit: Sequences and Series **Module:** Power Series Representations of Functions

Finding Power Series Representations by Integration

key concepts:

- You can do calculus on a power series inside its interval of convergence.
- The integral of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is $\sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C$.
- You can find the power series of new functions by applying calculus to known power series.

Integrating power series

Using calculus on power series is a powerful technique for finding alternative representations of functions.

example: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ Replace x with $-x$.

On The Side

$\int_0^x \frac{1}{1+t}\,dt = \ln(1+t)\Big|_0^x$ This is the definite integral of the function $\frac{1}{1+x}$.

$= \ln(1+t) - \ln(1+0)$

$= \ln(1+x)$, for $0 < x < 1$.

Here is a function represented by a geometric power series

Replacing x with $-x$ produces a new function and a new power series.

This is a definite integral that produces a function, not a value. Because one of the limits of integration is x, the integrand must be in terms of another variable. Usually that variable is t.

Choosing the interval (0, 1) keeps the argument of the natural log positive. You could even use (–1, 1) and $x + 1$ would be positive.

Integrating power series

Using calculus on power series is a powerful technique for finding alternative representations of functions.

example: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

$= 1 - x + x^2 - x^3 + \cdots$ Expand the series.

$\int_0^x \frac{1}{1+t}\,dt = \int_0^x (1 - t + t^2 - t^3 + \cdots)\,dt$ Integrate

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$

Integrating known power series results in alternative representations for new functions!

Take a look at the terms of the series so you can integrate it term by term.

Express the integrals as definite integrals in terms of t but with x as a limit of integration. Notice how this gets rid of the constant of integration C.

Express the terms of the integral using summation notation.

By integrating, you now have a power series representation of another function.

Unit: Sequences and Series **Module:** Power Series Representations of Functions

Integrating Functions Using Power Series

key concepts:

- If a function has a difficult integral, integrate its power series expression instead.
- The integral of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is $\sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C$.

Integrating the power series of e^{x^2}

Recall functions that seemed impossible to integrate.

The

allows you to find these integrals.

example: $\int_0^x e^{t^2} dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}$ Integrate term by term

re: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ So, $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

Note: $\int_0^x te^{t^2} dt = \frac{1}{2}\left(e^{x^2} - 1\right)$

You can find this answer by assuming that $u = t^2$ and $du = 2t\,dt$.

Not all functions can be integrated, but you can use power series to evaluate some integrals that you couldn't evaluate before.

This integral cannot be evaluated using *u*-substitution or integration by parts. So integrate the equivalent power series expression instead.

Because it is a definite integral with x as a limit of integration you don't have to add the constant.

Just make sure to form its power series by replacing x with x^2.

If there had been a factor of t in the integrand, you could have used *u*-substitution.

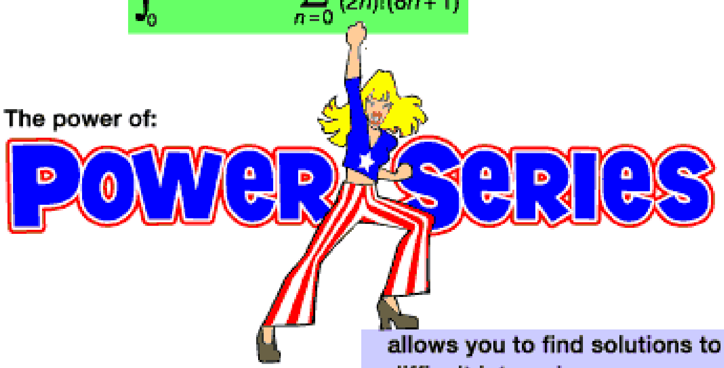
Integrating the power series of $\cos(t^4)$

example: $\int_0^x \cos(t^4)\,dt$

re: $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ So, $\cos(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n}}{(2n)!}$

$\int_0^x \cos(t^4)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+1}}{(2n)!(8n+1)}$ Integrate termwise

The power of:



allows you to find solutions to difficult integrals.

Here is another function that cannot be integrated by *u*-substitution. So integrate its power series representation.

Make sure to replace x by x^4 when you form its power series.

By integrating the general term of the power series you arrive at a power series expression of the integrand.

In this way, power series allow you to integrate functions you could not integrate with the other methods you have learned.

The disadvantage of using power series to integrate is that the result is not a recognizable function. Instead it's a power series. But if none of the other integration techniques work, that's a small price to pay.