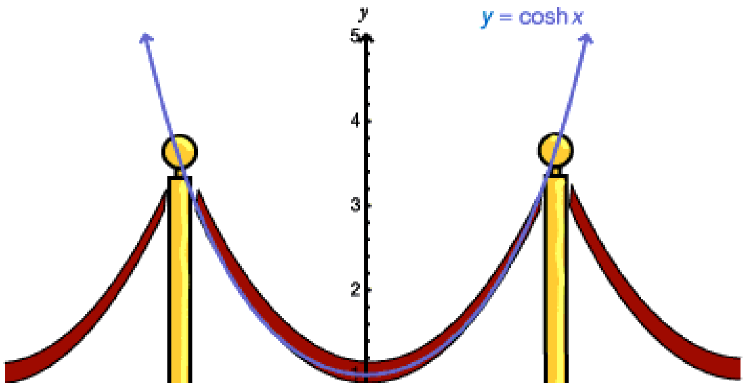
Unit: Elementary Functions and Their Inverses **Module:** The Hyperbolic Functions

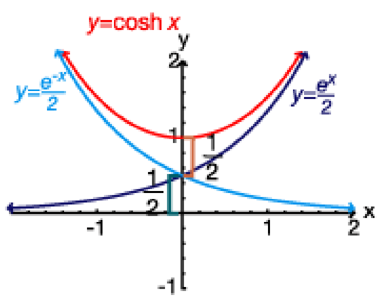
Defining the Hyperbolic Functions

key concepts:

- Hanging cables are called **catenary curves** and are described by hyperbolic cosine.
- There are six **hyperbolic functions** with names similar to those of the trig functions. They are defined by adding or subtracting **exponential functions**.



Cables, chains, ropes, and electrical wires – when hanging between two poles – are examples of **catenary curves**. These curves are described by the hyperbolic cosine function, cosh x.



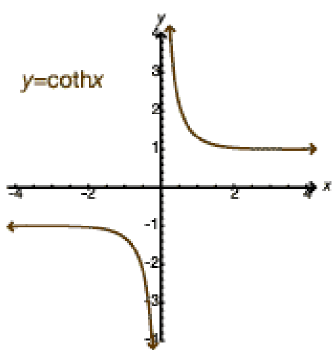
Featuring

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic cosine is actually defined as the sum of two **exponential functions**, $e^x/2$ and $e^{-x}/2$.

By studying the graphs of these two functions you can generate the graph of cosh x.

Right now it may not make sense why this function has a trigonometric name when it only involves exponentials. Professor Burger will discuss the reasons later.



Hyperbolic cotangent is defined in terms of hyperbolic sine and hyperbolic cosine.

Featuring

hyperbolic

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}},\ x \neq 0$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},\ x \neq 0$$

The "h" means that these are **hyperbolic functions**.

Remember that cosh x involves a sum while sinh x involves a difference.

The other four hyperbolic functions follow from the definitions of hyperbolic sine and hyperbolic cosine just like the trig functions are defined by sine and cosine.

Next up: Why do these functions have trigonometric-like names? And what's hyperbolic about them?

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Hyperbolic Identities

key concepts:

- When verifying a hyperbolic identity, use the definitions of the **hyperbolic functions**.
- The hyperbolic identities mirror many of the trigonometric identities
- The hyperbolic functions are derived from a hyperbola like the trig functions are derived from a circle.

Hyperbolic relationships

$$\cosh^2 x - \sinh^2 x = 1$$

Go back to the definitions.

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{\left(e^x + e^{-x}\right)^2}{4} - \frac{\left(e^x - e^{-x}\right)^2}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \qquad \text{FOIL.}$$

$$= \frac{\boxed{e^{2x}} + \boxed{2} + \boxed{e^{-2x}} - \left(\boxed{e^{2x}} \boxed{-2} + \boxed{e^{-2x}}\right)}{4}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

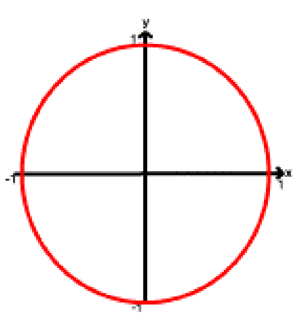
$$= \frac{4}{4} = \boxed{1}$$

Verify this hyperbolic identity by substituting the defining expressions of $\cosh x$ and $\sinh x$.

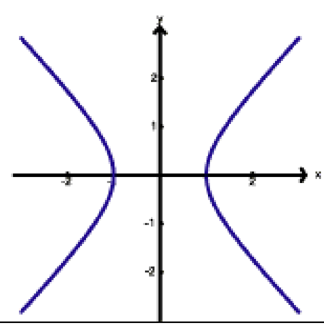
Since the numerators are binomials, you need to square them using FOIL or the distributive property. A classic mistake is to square just the terms of each binomial.

After combining terms and canceling, you can verify that the left-hand side of the identity does equal the right-hand side.

$$\cos^2 x + \sin^2 x = 1$$
$$X^2 + Y^2 = 1$$



$$\cosh^2 x - \sinh^2 x = 1$$
$$X^2 - Y^2 = 1$$



On the left, substituting X and Y for $\cos^2 x$ and $\sin^2 x$ illustrates their relationship to a circle.

Similarly, on the right, substituting X and Y for $\cosh^2 x$ and $\sinh^2 x$ illustrates their relationship to a hyperbola. Hence the name **hyperbolic function**.

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Derivatives of Hyperbolic Functions

key concepts:

- To differentiate the **hyperbolic functions**, use their definitions.
- The derivatives of the hyperbolic functions resemble those of the trigonometric functions.

Derivatives of hyperbolic functions

Find the derivatives of the hyperbolic functions by looking at their respective definitions.

Given $f(x) = \sinh(x)$, find $f'(x)$.

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\boxed{\cosh(x) = \frac{e^x + e^{-x}}{2}}$$

$$\begin{aligned}
\frac{d}{dx}[\sinh(x)] &= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] \\
&= \frac{1}{2}\cdot\frac{d}{dx}\left[e^x - e^{-x}\right] \\
&= \frac{1}{2}\left[e^x - e^{-x}(-1)\right] \\
&= \frac{1}{2}\left[e^x + e^{-x}\right] \\
&\boxed{= \frac{e^x + e^{-x}}{2} = \cosh(x)}
\end{aligned}$$

To determine the derivatives of the **hyperbolic functions**, you have to differentiate the exponential expressions that define them.

When you differentiate the expression for $\sinh x$ you produce the expression for $\cosh x$.

You don't have to go back to the definitions every time. After a while you will remember them.

$\frac{d}{dx}[\sinh(x)] = \cosh(x)$	$\frac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$
$\frac{d}{dx}[\cosh(x)] = \sinh(x)$	$\frac{d}{dx}[\operatorname{csch}(x)] = -\operatorname{csch}(x)\coth(x)$
$\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2(x)$	$\frac{d}{dx}[\coth(x)] = -\operatorname{csch}^2(x)$

Notice that the derivates of the hyperbolic functions are in some ways similar to those of the trigonometric functions. However, there are some differences.

The derivative of $\cosh x$ is $\sinh x$ even though the derivative of $\cos x$ is $-\sin x$.

And the derivative of $\operatorname{sech} x$ is $-\operatorname{sech} x \tanh x$ even though the derivative of $\sec x$ does not have a negative sign in front.

Hyperbolic functions and the chain rule

Given $f(x) = e^{\sinh x}$, find $f'(x)$.

$$\frac{d}{dx}[\sinh(x)] = \cosh(x)$$

$$\begin{aligned}
f'(x) &= e \cdot ()' && \text{Treat the exponent like a blop.} \\
&= e^{\sinh x}\cdot(\sinh x)' && \text{Multiply by the derivative of the blop.} \\
&= e^{\sinh x}\cdot(\cosh x) \\
&\boxed{= \cosh x \cdot e^{\sinh x}}
\end{aligned}$$

Here is an example that looks pretty mean. However, you only need the chain rule and one of the derivatives you just learned.

Notice that the exponent here is $\sinh x$. That's the inside part to which you will apply the chain rule.

The derivative of $\sinh x$ is $\cosh x$.

This answer has been regrouped for a nicer presentation.