Unit: Sequences and Series **Module:** The Ratio and Root Tests

The Ratio Test

key concepts:

- To apply the **ratio test** to a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

  If $\rho < 1$, then the series **converges absolutely**.
  If $\rho > 1$, then the series diverges.
  If $\rho = 1$, then the test is inconclusive.

The ratio test

Q: How do you test for absolute convergence?

The individual terms have to approach zero very quickly.

A: Use the ratio test.

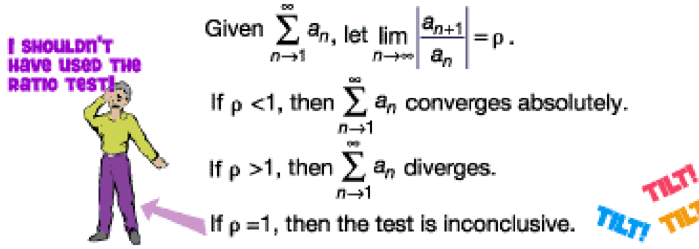
The ratio test

Given $\sum_{n\to 1}^{\infty} a_n$, let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$.

If $\rho < 1$, then $\sum_{n\to 1}^{\infty} a_n$ converges absolutely.

If $\rho > 1$, then $\sum_{n\to 1}^{\infty} a_n$ diverges.

If $\rho = 1$, then the test is inconclusive.



For a series to converge, its terms must decrease at a fast rate.

You can study this by using a ratio to compare two consecutive terms, a_{n+1} and a_n. As you take the limit of this ratio, you learn about the behavior of the series. In fact, since the **ratio test** involves the absolute values of the terms, you can determine **absolute convergence**, which is the stronger form.

If the limit of the ratio is less than one, then the terms are decreasing fast enough for the series to converge absolutely.

If the limit of the ratio is greater than one, then the series will diverge.

If the limit equals one, you cannot conclude anything. You must try a different test.

Using the ratio test

Example! $\sum_{n\to 1}^{\infty} \frac{1}{5^n}$

This is a geometric series with $r=1/5$. Since $r<1$, the series converges. The series has no negative values, so it converges absolutely.

Q: Can you use the ratio test to show that this series converges absolutely?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/5^{n+1}}{1/5^n}\right|$$

$$= \lim_{n\to\infty}\frac{5^n}{5^{n+1}} \qquad \text{Invert and multiply. Remove absolute value signs.}$$

$$= \lim_{n\to\infty}\frac{1}{5} \qquad \text{Cancel } 5^n.$$

$$= \frac{1}{5}$$

A: Yes, $\rho = \frac{1}{5} < 1$, so the series converges absolutely by the ratio test.

You know this series converges absolutely because it is a geometric series with a base that is positive and less than one. Will the ratio test agree that the series converges?

To apply the ratio test, you need to take the limit of the ratio of two consecutive terms a_{n+1} and a_n. Don't forget the absolute value symbol!

Since you have a fraction over a fraction, invert and multiply. The terms are already positive, so you can remove the absolute value symbol.

Canceling 5^n leaves one in the numerator and five in the denominator.

Use the Greek letter rho (ρ) to represent the value of the limit. Since ρ is less than one, the series converges absolutely.

Example! $\sum_{n\to 1}^{\infty} \frac{1}{n^2}$

Q: Can you use the ratio test to show that this series converges?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty} 1 = 1$$

A: No, $\rho = 1$, so the ratio test is inconclusive.

TILT!

Remember, if ρ equals one, you cannot conclude anything. Use another test. The series might converge like this p-series does, or it might diverge.

Unit: Sequences and Series **Module:** The Ratio and Root Tests

Examples of the Ratio Test

key concepts:

- Before testing a series for convergence, use the quicky test to make sure that the terms of the series approach zero.
- When you have a **factorial** in a series that you are testing for convergence, use the **ratio test**.

Using the ratio test on a divergent series

Example! $\sum_{n=1}^{\infty} \frac{4^n}{n^3}$

Use the quicky test to make sure that terms approach zero.

$$\lim_{n\to\infty} \frac{4^n}{n^3} = \infty$$

The quicky test shows that the series must diverge.

Q: Can you use the ratio test to show that this series diverges?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}/(n+1)^3}{4^n/n^3}\right|$$

Flip

$$= \lim_{n\to\infty} \frac{\cancel{4^{n+1}}\, n^3}{\cancel{4^n}\,(n+1)^3}$$

All terms are positive so remove the absolute value symbol. Invert and multiply.

$$= \lim_{n\to\infty} \frac{4n^3}{(n+1)^3}$$

Cancel.

$$= 4$$

The numerator is growing 4 times as fast as the denominator.

A: Yes, $\rho = 4 > 1$, so the series diverges by the ratio test.

The nth term test or quicky test is a simple test that can determine if a series is missing a condition for convergence. If the limit of the terms is not zero, the series will not converge. In this example, repeated applications of **L'Hôpital's rule** show that this limit of the terms is infinite, so the series diverges.

Will the **ratio test** produce the same result?

Take the limit of the absolute value of the ratio of two consecutive terms, a_{n+1} and a_n. You will need to invert and multiply. The terms are positive, so you can remove the absolute value symbol

After canceling, divide the top and the bottom by n^3, or use L'Hôpital's rule.

The limit is greater than one, so the series diverges by the ratio test..

The **factorial** $n!$ represents the product of the first n positive integers.

Example!

$$6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$$
$$4! = 4\cdot 3\cdot 2\cdot 1 = 24$$

The expression $n!$, read "n **factorial**", represents all the positive integers up to and including n multiplied together.

Factorial expressions increase very quickly.

Example! $\sum_{n=1}^{\infty} \frac{1}{n!}$

Q: Can you use the ratio test to show that this series converges?

Hint: When you see a factorial in a series, use the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)!}{1/n!}\right|$$

Don't make this mistake: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right|$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

All terms are positive so remove the absolute value symbol. Invert and multiply.

$$= \lim_{n\to\infty} \frac{\cancel{n}(\cancel{n-1})(\cancel{n-2})\cdots\cancel{2}\cdot\cancel{1}}{(n+1)\cancel{n}(\cancel{n-1})(\cancel{n-2})\cdots\cancel{2}\cdot 1}$$

$(n+1)! = (n+1)n!$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

Cancel.

$$= 0$$

The ratio test works really well with series that involve factorials.

Be careful when setting up the ratio. The a_{n+1} term goes on top. Its expression is $1/(n+1)!$, not $(n+1)!$.

Factorials involve lots of factors. In this case most of them cancel, which is why the ratio test is good for this type of series.

The limit is zero, which is less than one, so the series converges absolutely by the ratio test.

Unit: Sequences and Series **Module:** The Ratio and Root Tests

The Root Test

key concepts:

- To apply the **nth root test** to a series $\sum_{n=1}^{\infty} a_n$, let r = $\lim_{n\to\infty} \sqrt[n]{|a_n|}$.

  If $\rho < 1$, then the series **converges absolutely**.
  If $\rho > 1$, then the series diverges.
  If $\rho = 1$, then the test is inconclusive.

The root test

The n^{th} root test:

Idea behind the n^{th} root test: If you take the n^{th} root of a series and it approaches zero quickly, then the series itself approaches zero very quickly.

$$\text{Given } \sum_{n=1}^{\infty} a_n \text{, let } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho .$$

If $\rho<1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $\rho>1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\rho=1$, then the test is inconclusive.

The **nth root test** is another way to see if the terms of a series approach zero fast enough for the series to converge.

To apply the root test, evaluate the limit of the *n*th root of the absolute value of the general term a_n. Call the result rho (ρ).

Like the ratio test, a result less that one indicates that the series **converges absolutely**. A result greater than one means that the series diverges.

If the limit is one, then the test is inconclusive and you must try another test.

Using the root test

Example! $\sum_{n=1}^{\infty} \frac{1}{2^n}$

This is a geometric series with $r = 1/2$, so it converges. Since all its terms are positive, it converges absolutely.

Q: Can you use the n^{th} root test to show that this series converges?

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{2^n}\right|}$$

$$= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

This is a geometric series that you know converges. Will the root test agree?

Take the *n*th root of the absolute value of the general term a_n. In the denominator, the *n*th root of 2^n is 2, so the limit is 1/2.

Since 1/2 is less than one, the root test also shows that this series converges absolutely.

Example! $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^n}$

This series has a denominator that is growing very quickly. This series most likely converges.

Q: Can you use the n^{th} root test to show that this series converges?

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(n+1)^n}\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{(n+1)^n}} \quad \text{Take the absolute value.}$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)} \quad \text{Multiply exponents.}$$

$$= 0$$

A: Yes, $\rho = 0 < 1$, so the series converges absolutely by the n^{th} root test.

In this example you have an alternating series. It looks like the influential terms are in the denominator, so this series probably converges.

Since the top and bottom are both raised to the *n*th power, it is a good idea to try the *n*th root test.

The absolute value excludes the alternating $(-1)^n$.

The *n*th root cancels the *n*th power in the denominator, leaving just $(n + 1)$.

The limit is zero, so the series converges absolutely by the *n*th root test.