# Calculus Lecture Notes

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**Unit:** Applications of Integral Calculus **Module:** Arc Length

## Introduction to Arc Length

**key concepts:**

- **Arc length** is the length of the curve.
- The arc length of a smooth curve given by the function *f* (*x*) between *a* and *b*:

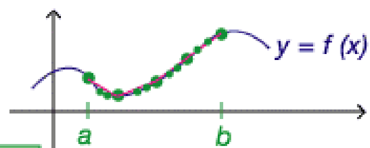
$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx\,.$$

### Breaking arc length into pieces

Plot some points and connect them with line segments.

Sum the distances between the points.

With more lines, the approximation gets better.

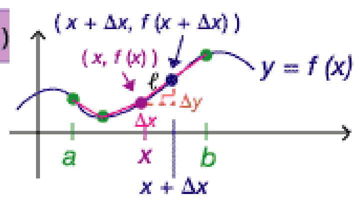



When measuring how long a line is, you can just use a ruler or the distance formula. But curves are trickier. It would be good to have a way to measure their lengths. This length is called **arc length**.

One way to think about arc length is to break a curve up into a lot of line segments. Then you can approximate the arc length by adding them all up.

### Deriving a formula for arc length

Consider the arbitrary point $(x, f(x))$




To find the exact arc length you have to use calculus.

Suppose the next point is really close. Call it $(x+\Delta x, f(x+\Delta x))$.

Use the Pythagorean theorem to determine the distance between the two points.

Label the legs of the triangle $\Delta x$ and $\Delta y$.

Let $\ell$ be the length of the line segment between the points.

Pick two points on the curve that are very close to each other. The second point is a small change in *x* from the first point.

The Pythagorean theorem tells you the length of the line segment connecting the two points.

$$\ell^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Notice that the length of the line segment is expressed in terms of the change in the two directions.

$$\text{Sum}_a^b \sqrt{(\Delta x)^2+(\Delta y)^2} = \text{Sum}_a^b \sqrt{(\Delta x)^2\left[1+\frac{(\Delta y)^2}{(\Delta x)^2}\right]}$$

Factor out $(\Delta x)^2$.

$$= \text{Sum}_a^b \,\Delta x\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}$$

Simplify.

$$= \text{Sum}_a^b \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

Change the order of the factors.

$$\text{Arc Length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\int_a^b \sqrt{1+[f'(x)]^2}\,dx}$$

To find the length of the entire curve, you must sum up the lengths of all the line segments.

Factoring out a $\Delta x$ moves the small change in *x* outside the radical sign.

If you let $\Delta x$ become arbitrarily small, then it acts like a *dx*. Then you can find the arc length by integrating.

Notice that the integral is different from the integral you would use to find the area under the curve.

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**Unit:** Applications of Integral Calculus **Module:** Arc Length

## Finding Arc Length

key concepts:

- The **arc length** of a smooth curve given by the function $f(x)$ between $a$ and $b$:

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx\,.$$

### *Applying the arc length formula*

Consider the function $f(x)=\dfrac{x^3}{6}+\dfrac{1}{2x}$ .

Find the arc length of this function between $x = 1$ and $x = 2$.

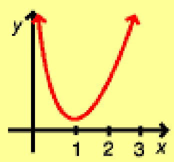
**Step: 1** Compute the derivative of $f$.

$$f'(x)=\frac{x^2}{2}-\frac{1}{2x^2}$$

**Step: 2** Insert the derivative of $f$ into the arc length formula.

Let $\ell$ be the arc length.

$$\ell=\int_1^2\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$




Given this curve, compute the **arc length** between $x = 1$ and $x = 2$.

To find the arc length, you will need the arc length formula. Notice that the arc length formula requires you to take the derivative. So do this first.

Once you have the derivative, put it into the formula with the correct limits of integration. The limits of integration are just the starting and ending $x$-values of the particular arc you want to measure.

**Step: 1** Compute the derivative of $f$. **Step: 2** Insert the derivative of $f$ into the arc length formula. **Step: 3** Evaluate the resulting integral.

$$\ell=\int_1^2\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$=\int_1^2\sqrt{1+\left(\frac{x^4-1}{2x^2}\right)^2}\,dx$$

$$=\int_1^2\sqrt{1+\frac{x^8-2x^4+1}{4x^4}}\,dx$$

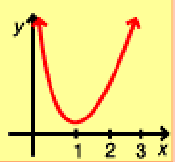
$$=\int_1^2\sqrt{\frac{4x^4+x^8-2x^4+1}{4x^4}}\,dx$$

$$=\int_1^2\sqrt{\frac{x^8+2x^4+1}{4x^4}}\,dx$$

$$=\int_1^2\sqrt{\frac{(x^4+1)(x^4+1)}{4x^4}}\,dx$$

$$=\int_1^2\sqrt{\frac{(x^4+1)^2}{(2x^2)^2}}\,dx$$

$$=\int_1^2\frac{x^4+1}{2x^2}\,dx$$ Simplify.




Now you have the integral and are ready to find the actual arc length.

Notice that in this example the integral is kind of difficult to evaluate. Integrals involving arc length tend to be more complicated than some other integrals because of the radical sign in the arc length formula. It's a good idea to look for a way to eliminate the radical sign.

Here the radical sign is cancelled by expressing the denominator and numerator in terms of a square.

$$\ell=\int_1^2\frac{x^4+1}{2x^2}\,dx$$

$$=\int_1^2\left(\frac{x^4}{2x^2}+\frac{1}{2x^2}\right)dx$$

$$=\int_1^2\left(\frac{x^2}{2}+\frac{1}{2}x^{-2}\right)dx$$

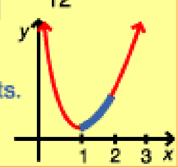
$$=\left(\frac{x^3}{6}-\frac{1}{2x}\right)\Bigg|_1^2$$

$$=\frac{2^3}{6}-\frac{1}{2(2)}-\left(\frac{1^3}{6}-\frac{1}{2(1)}\right)$$

$$=\frac{8}{6}-\frac{1}{4}-\left(\frac{1}{6}-\frac{1}{2}\right)$$

$$=\frac{7}{6}+\frac{1}{4}=\frac{14}{12}+\frac{3}{12}=\boxed{\frac{17}{12}}=1+\frac{5}{12}\approx 1.42$$

The length of the arc is approximately 1.42 units.




Start by breaking the fraction into two pieces. The integral is easier to evaluate that way.

Now the integral can be evaluated using the power rule.

Plugging in the limits of integration and simplifying give you the arc length.