Unit: Techniques of Integration **Module:** Integrals Involving Powers of Sine and Cosine

Introduction to Integrals with Powers of Sine and Cosine

key concepts:

- Use a **double-angle identity** to integrate $\sin^2 x$ or $\cos^2 x$.
- Use ***u*-substitution** to integrate $\sin^3 x \cos x$.

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x = 1 - \sin^2 x$$

Subtract $\sin^2 x$ from each side.

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$= 1 - \sin^2 x - \sin^2 x$$

$$\cos(2x) = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos(2x)$$

$$\boxed{\sin^2 x = \frac{1 - \cos(2x)}{2}}$$

Similarly

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

A **double-angle identity** is a trigonometric identity where the argument of the trigonometric function is multiplied by two.

The double-angle identities for sine and cosine can be derived from two simpler trigonometric identities. If you can remember those two identities, you can recreate the double-angle identities whenever you need them.

The double-angle identities are sometimes called **power-reducing identities** because they take the trig function down in power.

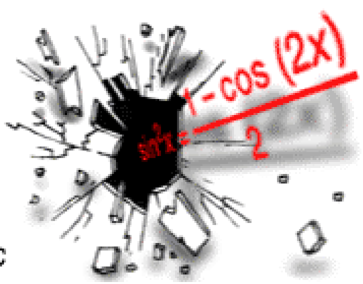
$$\int \sin^2 x\, dx = \int \frac{1 - \cos(2x)}{2}\, dx$$

$$= \int \left[\frac{1}{2} - \frac{\cos(2x)}{2}\right] dx$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos(2x)\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + C$$

$$\boxed{= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$$



To integrate $\sin^2 x$, start by applying the double-angle identity.

Next split the numerator into two terms. Do this so you can integrate terms piece-by-piece.

Integrating the first piece is straightforward. It's just a constant.

Integrating the second piece requires a minor ***u*-substitution**. Don't forget the constant of integration!

A similar method can be used to integrate $\cos^2 x$.

$$\int \sin^3 x \cos x\, dx = \int (\sin x)^3 \cos x\, dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x\, dx$$

Cosine is the derivative of sine.

$$= \int u^3\, du$$

Substitute u in place of x.

$$= \frac{u^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

Replace u with the original expression.

Integrating trigonometric expressions that contain both sines and cosines requires more finesse.

This particular expression can be solved with another *u*-substitution. Notice that by letting the new variable *u* equal the sine-term, the *du*-term contains the cosine-term. This substitution eliminates the extra cosine.

Once the substitution has been made, solving the integral just involves using the power rule.

The same method can be used to integrate any trigonometric expression of the form $\sin^n x \cos x$.

Unit: Techniques of Integration **Module:** Integrals Involving Powers of Sine and Cosine

Integrals with Powers of Sine and Cosine

key concepts:

- Sometimes ***u*-substitution** can't directly solve an integral made up of trig expressions.
- Integrate functions involving odd powers of sine or cosine by following three steps.
  1. Factor out a single sine or cosine from the odd power.
  2. Use the **Pythagorean identity** to convert the even power of sine to cosine or vice versa.
  3. Use *u*-substitution to evaluate the integral.

Q How can we find the antiderivative of $\sin^3 x$?

$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$$ Factor out $\sin x$.

$$= \int \left(1 - \cos^2 x\right)\sin x\,dx$$ Substitute $1 - \cos^2 x$ for $\sin^2 x$. Distribute.

$$= \int \left(\sin x - \cos^2 x \sin x\right)dx$$

$$= -\cos x - \int \cos^2 x \sin x\,dx$$

$$\int \cos^2 x \sin x\,dx = -\int u^2\,du = -\frac{u^3}{3} + C$$

Let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$ Solve for $\sin x\,dx$.

$$= -\cos x - \left[-\frac{(\cos x)^3}{3}\right] + C$$

$$= \boxed{-\cos x + \frac{\cos^3 x}{3} + C}$$

Some integrals have to be modified before you can use a ***u*-substitution**.

To solve this integral, start by factoring out a sine.

Convert the remaining factors into cosines using the **Pythagorean identity**.

Multiply through by the sine that was factored out.

The first piece of the integral is very easy to solve. The second piece is made up of cosines and a single sine. Now you can use *u*-substitution.

Let $u = \cos x$. Now you can use the power rule.

Make sure to leave your answer in terms of x and remember the constant of integration.

Remember

$\sin^2 x + \cos^2 x = 1$ the Pythagorean identity for sine and cosine

Here is the Pythagorean identity in terms of sine and cosine in case you have forgotten it.

Use the new method to evaluate $\int \sin^3 x \cos^3 x\,dx$.

$$\int \sin^3 x \cos^3 x\,dx = \int \sin^2 x \cos^3 x \sin x\,dx$$ Factor out $\sin x$.

$$= \int \left(1 - \cos^2 x\right)\cos^3 x \sin x\,dx$$ Replace $\sin^2 x$ with $1 - \cos^2 x$.

Let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$$= -\int \left(1 - u^2\right)u^3\,du$$

$$= -\int \left(u^3 - u^5\right)du$$ Distribute u^3 and add the exponents.

$$= -\left(\frac{u^4}{4} - \frac{u^6}{6}\right) + C$$

$$\boxed{= -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + C}$$

What if the integral is made up of sine and cosine?

Look for an odd power and factor out one of the terms. Here a sine term is factored out.

If you factored out a sine, use the Pythagorean identity to convert everything else to cosines. If you factored out a cosine, convert everything else to sine.

Now the integral can be solved by *u*-substitution. Notice that *du* will be made up of *dx* and the term you factored out.

Notice that integrals of this form simplify into sums and differences of trig functions. This happens because the Pythagorean identity converts the trig expression into a difference. Integrating may or may not introduce negatives into the expression.

Calculus Lecture Notes

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Unit: Techniques of Integration **Module:** Integrals Involving Powers of Trigonometric Functions

Integrals with Even and Odd Powers of Sine and Cosine

key concepts:

- Integrate functions involving odd powers of sine or cosine by following three steps.
  1. Factor out a single sine or cosine from the odd power.
  2. Use the Pythagorean identity to convert the even power of sine to cosine or vice versa.
  3. Use *u*-substitution to evaluate the integral.
- To evaluate integrals consisting only of even powers of sine or cosine, use the double-angle identities to reduce the degree of the integrand.

Integrating a higher power of sine

$$\int \sin^4 x\cos^5 x\,dx = \int \sin^4 x\cos^4 x\cos x\,dx$$

Factor out $\cos x$.

$$= \int \sin^4 x\left(\cos^2 x\right)^2\cos x\,dx$$

$$= \int \sin^4 x\left(1-\sin^2 x\right)^2\cos x\,dx$$

Use the Pythagorean identity.

Let $u = \sin x$

$du = \cos x\,dx$

$$= \int u^4\left(1-u^2\right)^2 du$$

$$= \int \left(u^4-2u^6+u^8\right)du$$

$$= \frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}+C$$

Don't forget the constant.

$$\boxed{= \frac{\sin^5 x}{5}-\frac{2\sin^7 x}{7}+\frac{\sin^9 x}{9}+C}$$

There is a great technique you can use when integrating expressions of sine and cosine where one of the trig functions is raised to an odd power.

Start by factoring out a single sine or cosine from the term with the odd power.

Then use the Pythagorean identity to express all the other even powers as the same trig function. The resulting form can be solved using a *u*-substitution.

Be careful with your algebra. Avoid careless mistakes by being thorough.

Remember to add the constant of integration to indefinite integrals.

Always leave your answers in terms of *x*.

Integrating even powers of sine and cosine

Q What do you do if all the powers of sine and cosine are even?

$$\int \sin^2 x\cos^2 x\,dx = \int\left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)dx$$

Use the double angle formulas.

$$= \frac{1}{4}\int\left(1-\cos^2(2x)\right)dx$$

$$= \frac{1}{4}x-\frac{1}{4}\int \cos^2(2x)\,dx$$

$$= \frac{1}{4}x-\frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx$$

Reapply the double angle formula.

$$= \frac{1}{4}x-\frac{1}{4}\left(\frac{1}{2}x+\frac{1}{2}\cdot\frac{1}{4}\sin(4x)\right)+C$$

$$\boxed{= \frac{1}{4}x-\frac{1}{4}\left(\frac{1}{2}x+\frac{1}{8}\sin(4x)\right)+C}$$

A Use double angle formulas to reduce the degree of the function until we can integrate it.

The above trick doesn't work if there is no odd power of sine or cosine. But there are other tricks.

You can use the double-angle formulas to simplify these integrals.

Move the constant outside the integral. Now break the integral into parts.

You can apply the double-angle formula again here. You can apply the formula as many times as you encounter a squared-term in the integrand.

NOTICE: When applying the double-angle formula again, the argument of the trig expression doubles. Make sure you are careful and remember to multiply the argument by two each time you use the formula.

Don't forget the constant of integration.

Remember

Double-angle formulas

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

Here are the double-angle formulas in case you have forgotten them.