Unit: Applications of Integration **Module:** Disks and Washers

Solids of Revolution

key concepts:

- Revolving a plane region about a line forms a **solid of revolution**.
- The volume of a solid of revolution using the **disk method** where $R(x)$ is the radius of the solid of revolution with respect to x is V where:

$$V = \pi \int_a^b [R(x)]^2 \, dx \, .$$

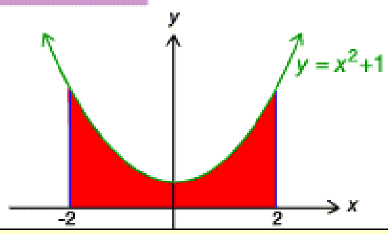
Revolving a region around an axis

Consider the region bounded by
the graph of $y = x^2+1$
the x-axis
the graph of $x = -2$
the graph of $x = 2$

Revolve this region around the x-axis to form a solid of revolution.

Q: Can you find the volume of that solid?

First graph the region.



Some solids can be described by moving regions through space as well as by slicing the solid into pieces.

Consider the plane region given to the left.

What happens if you rotate that region around the x-axis?

To visualize the solid, think of the region as though it were connected to the x-axis on a hinge. As the region moves through space around the hinge, the space it passes through makes up the solid.

A solid defined in this way is called a **solid of revolution**.

Finding volume using disks

The volume of the solid is the sum of the volumes of the disks.

$$V = \int_{-2}^{2} \text{Area of disk } dx$$

Now find the area of the disk.

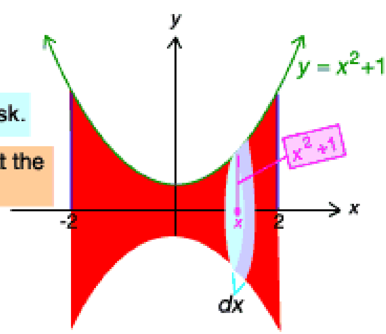
Now you will need the radius. At the point x, the radius is x^2+1.

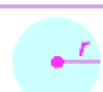
$$= \int_{-2}^{2} \pi (x^2+1)^2 \, dx$$

$$= 2\pi \int_0^2 (x^2+1)^2 \, dx$$

$$= 2\pi \int_0^2 (x^4+2x^2+1) \, dx$$

$$= \frac{412\pi}{15} \approx 86.289$$





Area of the disk:
$A = \pi r^2$

To find the volume of a solid of revolution, you can sometimes divide the region into slices. Each slice resembles a disk, so this method is called the **disk method**.

To find the volume, just integrate the areas of the disks across the given interval.

The radius of a given disk is equal to the height of the original region. The area of a disk equals the area of a circle.

Once you find the area, just integrate. Notice that sometimes you can find shortcuts in the integral based upon symmetry or other properties of the region.

Setting up the integral is the tough part of finding volumes. Once you have the integral, evaluating it is a piece of cake.

Unit: Applications of Integration **Module:** Disks and Washers

The Disk Method Along the y-Axis

key concepts:

- Revolving a plane region about a line forms a **solid of revolution**.
- The volume of a solid of revolution using the **disk method** where $R(y)$ is the radius of the solid of revolution with respect to y is V where:

$$V = \pi \int_a^b [R(y)]^2 \, dy .$$

Revolving a region about the y-axis

Consider the region bounded by:

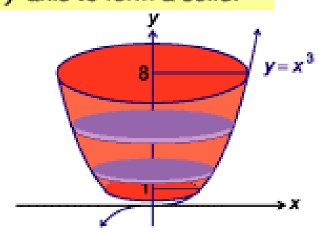
the curve of $y = x^3$

the y-axis

the graph of $y = 1$

the graph of $y = 8$

Revolve this region around the y-axis to form a solid.

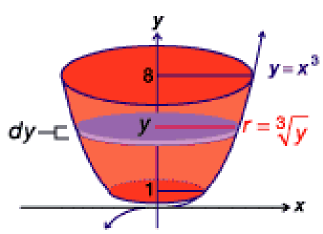


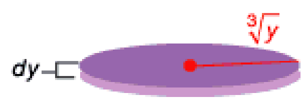
Q: How could you compute the volume of this solid?

A: Look for a way to slice the solid using simple shapes. Slicing vertically produces ill-defined shapes. Slicing horizontally produces circles.

In this example you are asked to find the volume of a **solid of revolution** rotated around the y-axis instead of the x-axis.

Notice that instead of slicing vertically like you would with a rotation about the x-axis, this solid seems easier to solve using the **disk method** with respect to the y-axis.





The area of the surface of the disk is πr^2, where the radius is x. Solve $y = x^3$ to express x in terms of the y-position.

$$y = x^3$$

$$\sqrt[3]{y} = \sqrt[3]{x^3}$$

$$x = \sqrt[3]{y}$$

$$V = \int_1^8 \pi(\sqrt[3]{y})^2 \, dy$$

$$= \int_1^8 \pi y^{2/3} \, dy$$

$$= \pi \int_1^8 y^{2/3} \, dy$$

$$= \pi \frac{3}{5} y^{5/3} \Big|_1^8$$

$$= \pi \left[\frac{3}{5}(8)^{5/3} - \frac{3}{5}(1)^{5/3} \right]$$

$$= \pi \left[\frac{3}{5}(32) - \frac{3}{5}(1) \right]$$

$$= \pi \left(\frac{96}{5} - \frac{3}{5} \right) = \frac{93\pi}{5}$$

Notice that the thickness of an arbitrary slice of the solid is given by a small change in y instead of a small change in x. That's a clue that you should integrate with respect to y for this problem.

Using the disk method to find the volume of vertically stacked disks is much like the method for horizontally stacked disks.

To integrate with respect to y you must express all of your functions in terms of y instead of x.

The disks span from $y = 1$ to $y = 8$, so these are the limits of integration. Notice that the limits of integration are also in terms of y.

The volume of each disk is equal to the area of the disk times the thickness. Remember, the area of a circle is given by the formula $A = \pi r^2$.

Integrate this product and the result is the volume of the solid of revolution.

Unit: Applications of Integration **Module:** Disks and Washers

A Transcendental Example of the Disk Method

key concepts:

- Revolving a plane region about a line forms a **solid of revolution**.
- The volume of a solid of revolution using the **disk method** where $R(x)$ is the radius of the solid of revolution with respect to x is V where:

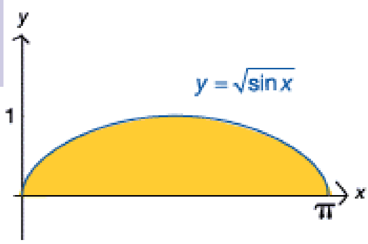
$$V = \pi\int_a^b [R(x)]^2\,dx\,.$$

A transcendental solid of revolution

Consider the region bounded by:

- the curve of $y = \sqrt{\sin x}$
- the x-axis
- the graph of $x = 0$
- the graph of $x = \pi$.

Revolve this region about the x-axis.



Look for a way to slice the solid using simple shapes. If you horizonally, you get complicated shapes. If you slice vertically, you get thin disks.

Find the volume of the **solid of revolution** described to the left.

Start by sketching the region in question. Notice that the sketch of the square root of sine looks like a distorted version of the sine curve.

Once you have the region you have to visualize what the solid of revolution will look like. Since you are revolving around the x-axis, the solid will look sort of like a football.

Slicing the solid horizontally doesn't look like a good idea. But vertical slices would give circular disks. Therefore you can find the volume of the solid by using the **disk method**.

Finding the volume of the transcendental solid

Find the volume using the disk method.

$$V = \int_0^{\pi} \pi\left(\sqrt{\sin x}\right)^2 dx$$

$$= \int_0^{\pi} \pi \sin x\,dx$$

$$= \pi\int_0^{\pi} \pi \sin x\,dx$$

$$= -\pi\cos x\Big|_0^{\pi}$$

$$= -\pi\cos\pi - (-\pi\cos 0)$$

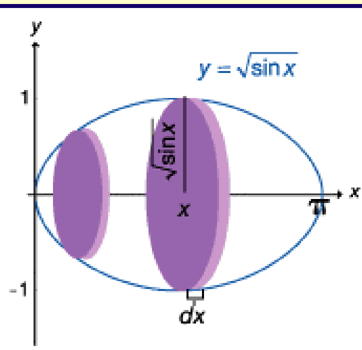
$$= -\pi(-1) - (-\pi\cdot 1)$$

$$= \pi - (-\pi)$$

$$= \pi + \pi$$

$$= 2\pi$$

The volume of the solid is 2π units cubed.



The thickness of each disk is a small value in the x-direction. Call it dx. The disks run from 0 to π, which are the limits of integration.

The radius of each disk is equal to the y-value of the function. Square the radius and multiply by π.

Simplify the product so it is easier to integrate.

Notice that the hardest step in these volume problems is setting up the integral. The actual calculus is not that tough. It's the set-up that is a little confusing.

Unit: Applications of Integration **Module:** Disks and Washers

The Washer Method across the x-Axis

key concepts:

- The volume of a **solid of revolution** using the **disk method** where $R(x)$ is the radius of the solid of revolution with respect to x is V where:

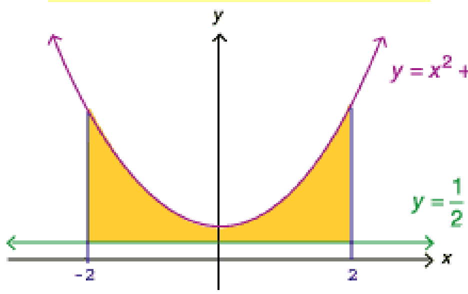
$$V = \pi \int_a^b [R(x)]^2 \, dx .$$

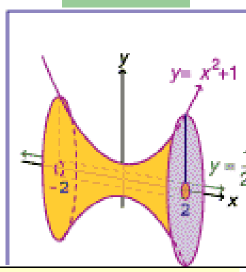
- **Washers** are disks with smaller disks removed from the center.

A new type of solid of revolution

Consider the region bounded by the curves of: $y = x^2 + 1$, $y = \frac{1}{2}$, $x = -2$, $x = 2$

Revolve this region about the x-axis.





Suppose you are asked to find the volume of the **solid of revolution** described to the left.

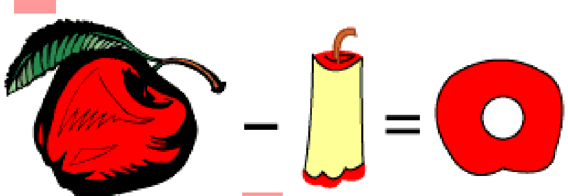
The first step would be to graph the region in question. The region is bound by a parabola and three lines.

Notice that the region is not flush with the axis of revolution. This means that there will be a hole in the solid of revolution.

How can you calculate the volume of a solid of revolution with a hole in it?

Coring an apple and slicing it

Q: If you slice a cored apple, will you get disks?



A: No, you get a disk with a disk cut out of it.

Whenever you encounter a difficult problem, it is a good idea to consider a simpler problem first.

Consider an apple that you have cored out the center. What shape does the cross-section take when you slice the solid?

Each slice looks like a regular apple slice, but with a hole in the center. These slices are called **washers**. The area of each piece is easy to calculate, since all you have to do is subtract the area of the inner circle from the area of the outer.

Finding volume using washers

Q: What is the volume of the solid?

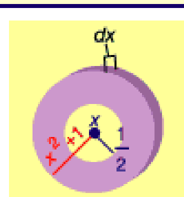
$$V = \int_{-2}^{2} \left[\pi (x^2 + 1)^2 - \pi \left(\frac{1}{2} \right)^2 \right] dx$$

Subtract the area of the inner disk.

$$= \int_{-2}^{2} \left[\pi (x^2 + 1)^2 - \pi \left(\frac{1}{4} \right) \right] dx$$

$$= 2\pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x - \frac{1}{4} x \right] \Bigg|_0^2 = 2\pi \left[\frac{2^5}{5} + \frac{2(2)^3}{3} + 2 - \frac{1}{4}(2) - 0 \right]$$

$$= 2\pi \left[\frac{32}{5} + \frac{16}{3} + 2 - \frac{1}{2} \right] = 2\pi \left[\frac{192}{30} + \frac{160}{30} + \frac{60}{30} - \frac{15}{30} \right] = 2\pi \frac{397}{30} = \boxed{\frac{397}{15}} \pi$$



Finding volumes using washers is like finding volumes using the **disk method**. But instead of integrating the whole area of the disk, you only integrate the area of the washer. Remember, the area of the washer equals the area given by the outer radius minus the area given by the inner radius.

Now you can find the volume of solids of revolution with holes in them.

Unit: Applications of Integration **Module:** Disks and Washers

The Washer Method across the y-Axis

key concepts:

- The volume of a **solid of revolution** using the **disk method** where $R(y)$ is the radius of the solid of revolution with respect to y is V where:

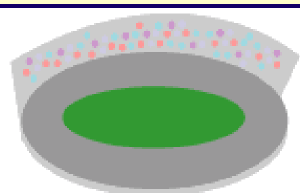
$$V = \pi\int_a^b [R(y)]^2\,dy\,.$$

- **Washers** are disks with smaller disks removed from the center.

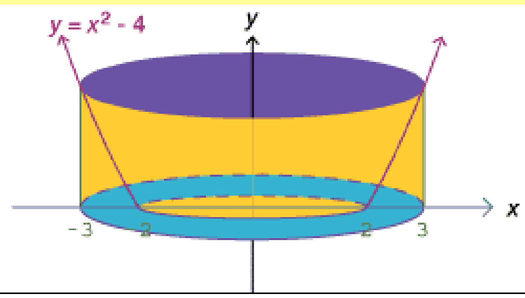
Volume of a racetrack

The boundaries of the racetrack are:

the graph of $y = x^2 - 4$
the x-axis
the graph of $x = 2$
the graph of $x = 3$



This region revolves around the y-axis to form the racetrack.



Suppose you are given a **solid of revolution** defined by the four curves to the left. What is the volume of this solid of revolution?

Start by graphing the four curves. The region described by the curves is a small piece that looks like a triangle someone sat on.

Rotating the region across the y-axis gives you the solid of revolution.

Slicing the solid vertically gives very strange shapes. But horizontal slices give **washers**. So you should try to find the volume using the washer method.

Summing the washers

$$V = \int_0^5 \left[\pi(9) - \pi\left(\sqrt{y+4}\right)^2\right] dy$$

The outer radius is three. $(3)^2 = 9$

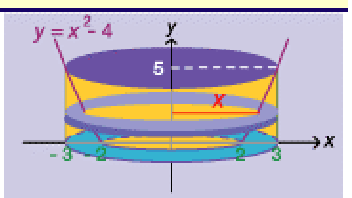
The inner radius is $\sqrt{y+4}$

$$= \int_0^5 \left[\pi(9) - \pi(y+4)\right] dy$$

$$= \pi\left[9y - \frac{y^2}{2} - 4y\right]_0^5$$

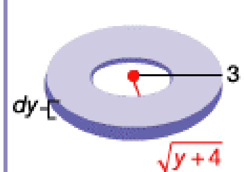
$$= \pi\left[45 - \frac{25}{2} - 20\right]$$

$$= \frac{25\pi}{2}$$



The boundaries of the racetrack are:

the graph of $y = x^2 - 4$
the x-axis $(y = 0)$
the graph of $x = 2$
the graph of $x = 3$



Q: Is there a simpler way to slice the solid?

Remember, the washer method is just a modification of the **disk method**. Instead of integrating the entire area, you only integrate the area of the washer. To do that, you must subtract the area given by the inner radius from the area given by the outer radius.

Since the thickness of each disk is a little change in y, your integral needs to be expressed in terms of y. That means the limits of integration and the radius must all be in terms of y instead of x.

It is a great idea to draw a small washer so you can better visualize which equations give which radii.