Unit: Elementary Functions and Their Inverses **Module:** Inverse Trigonometric Functions

The Inverse Sine, Cosine, and Tangent Functions

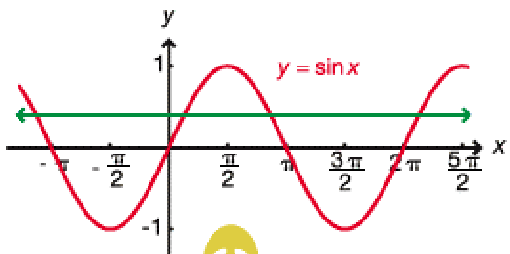
key concepts:

- The standard trigonometric functions do not have **inverses**. Only by restricting the domain can you make them **one-to-one** functions.
- The inverse trig functions can be indicated by a raised –1 or by the prefix "arc".

Trigonometric inverses

Consider the standard trigonometric functions.
Are they invertible?

Check the graph of one and see.



Does sine pass the horizontal line test?

NO

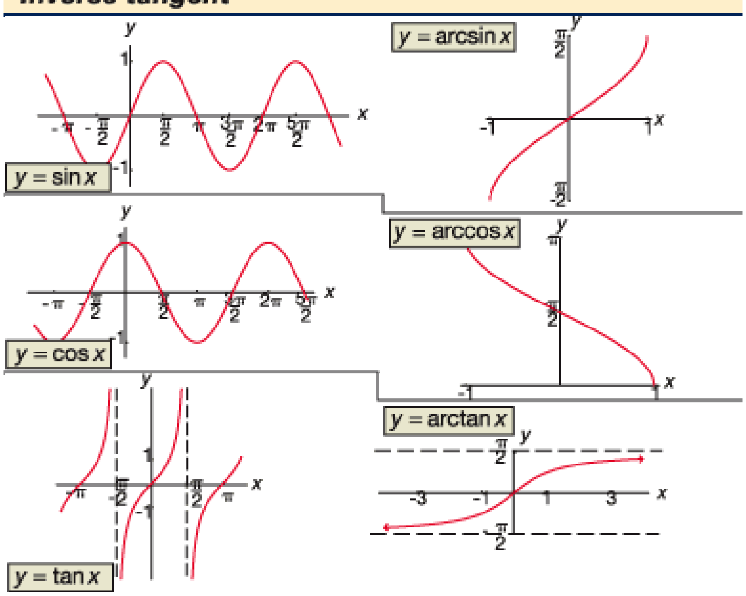
The sine, cosine and tangent functions do not pass the **horizontal line test**. Therefore they do not have **inverses**.

Despite that fact, it would be useful to find a way to define inverse trigonometric functions.

Notice that from $-\pi/2$ to $\pi/2$ the sine function is increasing. On this restricted domain sine is **one-to-one**.

You can find other domains where sine is also increasing and therefore one-to-one. You can even find domains where it is decreasing. The sine function will be one-to-one there, too. You can define lots of inverses for the sine function. However, mathematicians established a convention to use $[-\pi/2, \pi/2]$ for the standard inverse sine function.

Inverse tangent



Here are the graphs of the inverses of sine, cosine, and tangent. They are labeled **arcsine**, **arccosine**, and **arctangent**, respectively, instead of using the –1 notation.

Notice that arccosine is defined by a different interval than the others. The cosine function is restricted to the interval $[0, \pi]$ in order to define its inverse.

Tangent is restricted to $[-\pi/2, \pi/2]$, just like sine. The vertical asymptotes for tangent are translated into horizontal asymptotes for arctangent.

Calculus Lecture Notes

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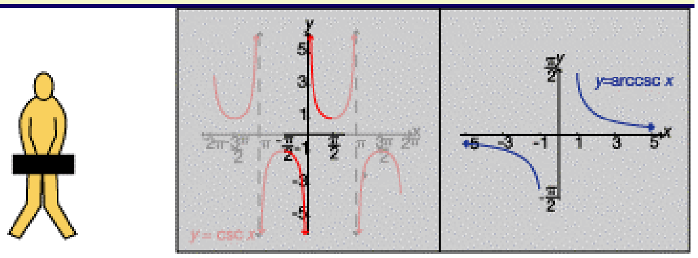
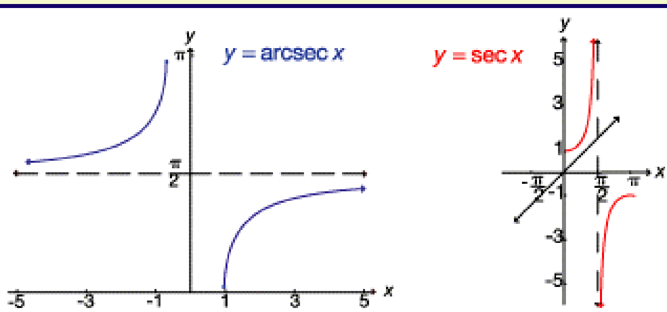
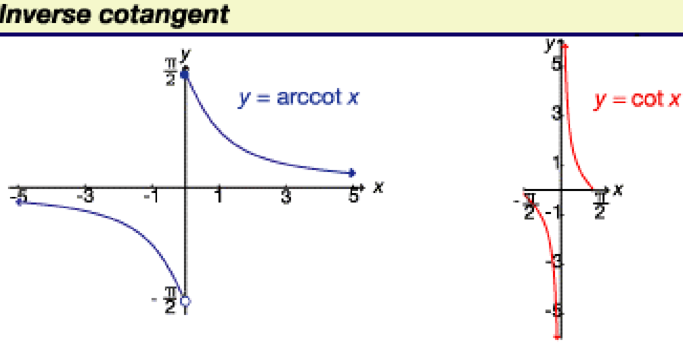
Unit: Elementary Functions and Their Inverses **Module:** Inverse Trigonometric Functions

The Inverse Secant, Cosecant, and Cotangent Functions

key concepts:

- The standard trigonometric functions do not have **inverses**. Only by restricting the domain can you make them **one-to-one** functions.

$\frac{1}{\sin x} = \csc x \quad \frac{1}{\cos x} = \sec x \quad \frac{1}{\tan x} = \cot x$	The cosecant, secant, and cotangent functions are reciprocals of the sine, cosine, and cotangent functions. Don't confuse reciprocal with **inverse**.
Inverse cosecant  	Before defining an inverse for cosecant, you must restrict its domain. The convention is to restrict it to the interval [–π/2, π/2]. Since cosecant is not defined at zero, arccosecant never equals zero. It ranges from –π/2 to π/2, skipping zero.
Inverse secant  	Like cosine, secant must be restricted to the interval [0, π] in order to be invertible. Reflecting the graph of secant across the line defined by $y = x$ produces the graph of arcsecant. Notice that arcsecant never equals π/2, since secant is not defined there.
Inverse cotangent  The inverse cotangent is asymptotic to the *x*-axis. In order to satisfy the definition of a function, modify the range to $-\frac{\pi}{2} < y \le \frac{\pi}{2}, y \ne 0$.	For cotangent, the convention is to restrict the domain to (–π/2, π/2]. Notice that this is a half-open interval, because cotangent is equal to zero at both –π/2 and π/2. You don't want arccotangent to have both of those values at zero. Notice that arccotangent never equals zero, since cotangent is not defined there.

Evaluating Inverse Trigonometric Functions

key concepts:

- To evaluate inverse trigonometric expressions, first convert them into standard trig expressions. Use this technique to solve inverse trig equations as well.
- An **inverse trig function** will not reverse the original function outside of the domain of the inverse trig functions,

Evaluating inverse trig functions by hand

Evaluate $\arcsin\left(\frac{1}{2}\right)$.

Start by thinking backwards.

$\sin x = \frac{1}{2}$ What is the angle whose sine is $\frac{1}{2}$?

$\sin\frac{\pi}{6} = \frac{1}{2}$ Remember the allowed domain. $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

The key to evaluating inverse trigonometric expressions is to convert them into standard trig expressions.

To find arcsin(1/2), you need to look for “the angle whose sine is 1/2". In other words, look for x such that $\sin x = 1/2$.

Of course, there are lots of angles for which the sine is 1/2. But only one of them lies in the restricted domain for sine that you used to define arcsine.

Evaluate arctan (1).

Think backwards.

$\tan x = 1$ What is the angle whose tangent is 1?

$\tan\frac{\pi}{4} = 1$ The angle must be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$\arctan(1) = \frac{\pi}{4}$

There are lots of angles for which tangent is one, but only one of them lies in the restricted domain. That’s what makes arctangent a function.

Evaluating inverse trig functions with a graphing calculator

Evaluate arctan (1).

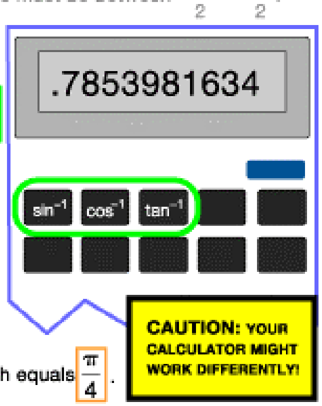
Think backwards.

$\tan x = 1$ What is the angle whose tangent is 1?

$\tan\frac{\pi}{4} = 1$ The angle must be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$\arctan(1) = \boxed{\frac{\pi}{4}}$

Make sure your calculator is in radian mode.



First press the $\tan^{-1}$ key.

Next press the 1.

Close the parentheses.

Then press ENTER.

The calculator gives .7853981634 which equals $\frac{\pi}{4}$.

Most scientific calculators have buttons for evaluating inverse trigonometric functions. The buttons usually have the raised –1 notation.

Before you start, decide whether you want the answer in degrees or radians. Calculus usually prefers radians.

To evaluate arctan(1), you will need to press the $\tan^{-1}$ key. For some calculators you will need to enter 1 and then press the $\tan^{-1}$ key. Graphing calculators like Professor Burger’s usually require you to press $\tan^{-1}$ and then 1.

Even though the calculator is in radian mode, it usually presents its answer as a decimal.

Of course, your calculator may work differently, so check the user’s manual.

Evaluating Inverse Trigonometric Functions

Special behavior of inverse trig functions

Evaluate $\arcsin(\sin(\pi))$.

Now think backwards.

$$\arcsin(\sin(\pi)) = \arcsin(0)$$

Simplify the trig expression.

What is the angle whose sine is 0?

$$\sin x = 0$$
$$\sin 0 = 0$$
$$\arcsin(\sin(\pi)) = 0$$

Notice that the inverse functions didn't cancel each other out.

Q: What went wrong?

A: Nothing! The inverse trig functions are defined only on specific intervals.

$\arcsin(\sin(x))$ does not always equal x.

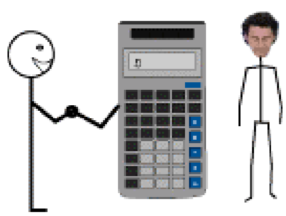
To evaluate arcsin(sin(π)), you need to evaluate the inside first. Then the expression becomes arcsin(0), which you can transform to look for "the angle whose sine is zero."

So the value of arcsin(sin(π)) is zero.

Wait a minute! The inverse function machine is supposed to give back what the original machine started with. The two functions should cancel and produce π!

This is why the intervals are important. Arcsine is only defined from –π/2 to π/2. If you start with a number in this interval, then the functions will cancel each other.

Use the calculator to check.



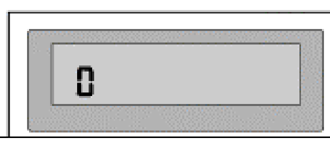
First press the $\boxed{\sin^{-1}}$ key.

Next press the $\boxed{\sin}$ key.

Now press the $\boxed{\pi}$ key.

Close the parentheses.

Then press $\boxed{\text{ENTER}}$ key.



The calculator agrees with Dr. Burger.

Remember, if you have a different calculator, you may have to press the buttons in the reverse order.

The calculator cannot think, it can only calculate. It may produce inaccurate results if it is not in the correct mode. Make sure to evaluate the expression based on your knowledge of the functions before using the calculator to check.

Solving equations with inverse trig functions

Given $\arctan(2x-3) = \frac{\pi}{4}$, solve for x.

Think backwards.

$\frac{\pi}{4}$ is the angle whose tangent is $2x-3$.

$$\tan\left(\frac{\pi}{4}\right) = 2x - 3$$

Simplify the tangent.

$$1 = 2x - 3$$

Solve for x.

$$4 = 2x$$
$$\boxed{2 = x}$$

You can solve inverse trig equations using the same method you used to evaluate inverse trig expressions.

First think backwards by expressing the equation in terms of standard trig functions.

Evaluate the trig expression and solve like any other equation.

Check your work by plugging the solution back in to the original equation to verify that it is a solution.