Unit: Applications of Integration **Module:** Work

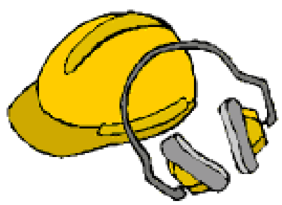
Introduction to Work

key concepts:

- **Work** is the energy used when applying a force over a distance.
- For a constant force F, work is the product of the force and the change in distance.
- For a changing or variable force $F(x)$ on $[a,b]$, work is given by the integral:

$$\int_a^b F(x)\,dx\ .$$

Forces at work



Most people have a good idea of what **work** means to them, but in physics and mathematics work has a very specific meaning.

Forces at work

For a constant force F, work is the product of the force and the change in distance.

$$W = F \cdot \Delta D$$
$$W = F\Delta x$$

If you apply a constant force of 20 lb over a distance of five feet, then

$$W = 20\text{ lb} \cdot 5\text{ ft}$$
$$W = 100\text{ lb} \cdot \text{ft}$$

Work is the energy used when moving an object. For work to be done a force must be applied to an object and that object must move in the direction of the force. So pushing against a wall might take a lot of effort, but doesn't do any work.

When a force is constant, work can be found by multiplying the magnitude of the force by the change in distance.

Consider a changing force $F(x)$ between two points a and b.

$$W = \int_a^b F(x)\,dx$$

Work is the energy used when applying a force to an object over a distance.

$$W = \int_a^b F(x)\,dx$$

But forces do not have to be constant. To find the work done using a variable or changing force you have to use calculus. $F(x)$ means the force at a particular x-value. It's a notational way of saying "changing force."

Notice that the integral looks a lot like the formula for a constant force. The force is multiplied by a change in distance, this time dx. The integral sums up all the infinite little pieces.

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Calculating Work

key concepts:

- **Work** is the energy used when applying a force over a distance.
- For a changing or variable force $F(x)$ on $[a, b]$, work is given by the integral:

$$\int_a^b F(x)\,dx \text{ .}$$

Measuring work

For a constant force F, work is the product of the magnitude of the force and the distance over which it has been applied.

The units for work are force units times distance units.

If the force is in pounds (lb), and the distance is in feet (ft), then the units are foot-pounds (ft-lb).

For a constant force, $W = F \cdot \Delta D$

For a changing force $F(x)$ between a and b, $W = \int_a^b F(x)\,dx$.

Work is the energy used to move an object. If the object doesn't move in the direction of the force, then no work is done.

Finding the work done by a constant force is pretty simple. You just multiply the magnitude of the force by the change in distance.

Work is measured in force units times distance units. Be careful when talking about work. You have to make sure you are using the right units.

For a variable force, work is computed using a definite integral.

Moving a refrigerator

Example Consider Prof. Burger moving a refrigerator starting 1ft from a wall and ending 30 ft from it.

The force Prof. Burger uses to move the refrigerator is given by $F(x) = \frac{4}{x}$ lb.

Q: What work is done as Prof. Burger moves the refrigerator from 1ft to 30ft?

$$W = \int_1^{30} \frac{4}{x}\,dx$$
$$= 4\ln|x| \Big|_1^{30}$$
$$= 4\ln 30 - 4\ln 1$$
$$= 4\ln 30 - 0$$
$$= 4\ln 30 \text{ ft-lb}$$
$$\approx 13.604 \text{ ft-lb}$$

A: Prof. Burger performed $4\ln 30$ ft-lb of work.

NOTE: Although there is nothing incorrect about expressing the units as lb-ft, the convention is ft-lb.

Consider this example. Remember that if Prof. Burger didn't move the refrigerator he would have done no work no matter how hard he pushed.

The force Prof. Burger is using to move the refrigerator is a variable force defined in terms of x.

To calculate the work, plug the force and the positions into the work formula.

Remember, the natural log of one is zero.

The force unit is the pound (lb). The distance unit is the foot (ft). So the work unit is the foot-pound (ft-lb).

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Unit: Applications of Integration **Module:** Work

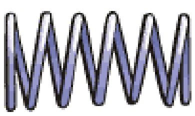
Hooke's Law

key concepts:

- **Work** is the energy used when applying a force over a distance.
- Use **Hooke's law** to determine how much force it takes to stretch a spring.

Stretching a spring

The more you stretch a spring, the more force it takes to stretch it.



Easy to stretch

Hard to stretch

When you stretch out a spring, the further you stretch it the harder it gets.

Hooke's law

Hooke's Law

0

$$F(x) = kx$$

You can use **Hooke's law** to determine how much force it takes to stretch a spring.

Springs all have a resting length. The resting length is how long the spring is before you stretch it.

The variable x represents the spring's displacement from its resting length. If the spring is 3 inches long at rest and you stretch it to 5 inches, then you have displaced it 2 inches.

The constant k is called the spring constant. Each spring has its own spring constant. Often you will have to determine this value from information you are given.

Using Hooke's law

Example

$F(x) = kx$

Suppose you have the following information about a spring.

The resting length is two inches.

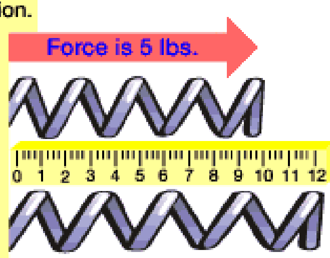
The force required to stretch the spring 10 inches is five pounds.

Q: How much work is done by stretching the spring 12 inches?

$$5 = F(10)$$
$$5 = k \cdot 10$$
$$k = \frac{1}{2}$$
$$F(x) = \frac{1}{2}x$$

Integrate the force function.

$$\int_0^{12} \frac{1}{2} x\,dx = \left.\frac{x^2}{4}\right|_0^{12}$$
$$= \frac{144}{4} - 0$$
$$= \frac{72}{2}$$
$$= \boxed{36 \text{ in-lb}}$$



A: The work performed is 36 in-lb.

Here is an example of Hooke's law in action.

Suppose you are given a spring whose resting length is two inches and you know that it takes five pounds to stretch the spring ten inches.

To calculate the **work** done by stretching the spring you first have to determine the spring constant.

You know that to displace the spring ten inches requires a five-pound force. Plug those values into Hooke's Law to find the spring constant.

Once you have the spring constant you can integrate the force equation to determine the work required to stretch the spring any length.

Here the spring is stretched twelve inches. So integrate from zero to twelve.