Unit: L'Hôpital's Rule **Module:** Other Indeterminate Forms

L'Hôpital's Rule and Indeterminate Products

key concepts:

- Some **indeterminate forms** have to be transformed before you can apply **L'Hôpital's rule**.
- When applying L'Hôpital's rule to an indeterminate product, express one of the factors as a fraction.

Camouflaged indeterminate forms

Evaluate: $\lim_{x\to\infty} e^{-x}\ln x$

$\lim_{x\to\infty} e^{-x}\ln x \to 0\cdot\infty$

This is a camouflaged indeterminate form, or indeterminate product.

An example of a camouflaged **indeterminate form** is the indeterminate product $0\cdot\infty$. It is indeterminate because you cannot tell who wins. Zero times anything is zero, but anything times infinity is infinity, so what is the limit?

The indeterminate form 0 • ∞

Evaluate: $\lim_{x\to\infty} e^{-x}\ln x$

$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{1}{e^x}\cdot\ln x$ Rewrite the indeterminate product.

NOTE: $e^{-x} = \frac{1}{e^x}$

$= \lim_{x\to\infty} \frac{\ln x}{e^x} \to \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{e^x}$ Use L'Hôpital's Rule.

$= \lim_{x\to\infty} \frac{1}{xe^x} = 0$

If you write e^{-x} as $\frac{1}{e^x}$, then your limit produces the standard indeterminate quotient $\frac{\infty}{\infty}$, which allows you to use **L'Hôpital's rule**.

Evaluate: $\lim_{\theta\to 0} \theta\cdot\cot\theta$

Theta (θ) is just another variable. Don't let it confuse you.

$\lim_{\theta\to 0} \theta\cdot\cot\theta = \lim_{\theta\to 0} \theta\cdot\frac{1}{\tan\theta}$

You can think of ∞ as $\frac{1}{0}$.

NOTE: $\cot\theta = \frac{1}{\tan\theta}$

$= \lim_{\theta\to 0} \frac{\theta}{\tan\theta}$

$= \lim_{\theta\to 0} \frac{1}{\sec^2\theta}$ Apply L'Hôpital's Rule.

$= \lim_{\theta\to 0} \frac{1}{\left(\frac{1}{\cos^2\theta}\right)}$

$= \cos^2 0 = 1$

This limit also produces the indeterminate product $0\cdot\infty$.

Here it makes sense to write $\cot\theta$ as the reciprocal of $\tan\theta$. Then you have the other standard indeterminate quotient, $0/0$.

L'Hôpital's Rule and Indeterminate Differences

key concepts:

- Some **indeterminate forms** have to be transformed before you can apply **L'Hôpital's rule**.
- Look for a common denominator or a clever way of factoring in order to transform an indeterminate difference into an indeterminate quotient so you can apply L'Hôpital's rule.

Indeterminate differences

Evaluate: $\lim\limits_{x\to 0}\frac{1}{x}-\frac{1}{e^x-1}$

$$\lim_{x\to 0}\frac{1}{x}-\frac{1}{e^x-1} = \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)} \to \frac{0}{0}$$

Use L'Hôpital's rule.

$$= \lim_{x\to 0}\frac{e^x-1}{xe^x+e^x-1} \to \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{e^x}{xe^x+e^x+e^x}$$

$$= \lim_{x\to 0}\frac{e^0}{(0)e^0+e^0+e^0} = \frac{1}{0+1+1} = \boxed{\frac{1}{2}}$$

This is an example of an indeterminate difference that you can transform by finding a common denominator.

Once you have expressed the limit as quotient, it produces the standard **indeterminate form** $0/0$.

A second application of **L'Hôpital's rule** is needed since the limit produces an indeterminate form again.

Evaluate: $\lim\limits_{x\to\infty}\sqrt{4x^2+x}-2x$

Tip: Look for sneaky ways to factor.

$$\lim_{x\to\infty}\sqrt{4x^2+x}-2x = \lim_{x\to\infty}\sqrt{x^2\left(4+\frac{1}{x}\right)}-2x$$

Factor out an x-squared.

$$= \lim_{x\to\infty} x\sqrt{4+\frac{1}{x}}-2x$$

$$= \lim_{x\to\infty} x\left(\sqrt{4+\frac{1}{x}}-2\right)$$

$$= \lim_{x\to\infty}\frac{\sqrt{4+\frac{1}{x}}-2}{\left(\frac{1}{x}\right)} \to \frac{0}{0}$$

$$= \lim_{x\to\infty}\frac{\left[\frac{1}{\left(2\sqrt{4+\left(\frac{1}{x}\right)}\right)}\right]\cdot\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \lim_{x\to\infty}\frac{1}{2\sqrt{4+\left(\frac{1}{x}\right)}}$$

$$= \frac{1}{2\sqrt{4+(0)}} = \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \boxed{\frac{1}{4}}$$

This limit produces an indeterminate difference, but it's not obvious how to find a common denominator.

Try factoring the expression, being very careful when working under the radical.

Once you have factored out x, you can send it to the denominator by finding it's reciprocal, $\frac{1}{x}$.

Now you have a limit that produces the form $\frac{0}{0}$, so you can apply L'Hôpital's rule.

The numerator includes a square-root expression, so you'll have to use the chain rule.

Cancel common factors and plug in the value to determine the limit.

Calculus Lecture Notes

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Unit: L'Hôpital's Rule **Module:** Other Indeterminate Forms

L'Hôpital's Rule and One to the Infinite Power

key concepts:

- Some **indeterminate forms** have to be transformed before you can apply **L'Hôpital's rule**.
- In order to apply L'Hôpital's rule to a limit of the form 1^{∞} use the properties of logarithms to rewrite the exponent as a logarithm.

Even more camouflaged limits

Evaluate: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x}$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

This expression is the key to solving for the limit.

$$e^{\ln(\text{anything})} = (\text{anything})$$

You may encounter a limit that produces one to the infinite power, 1^{∞}, which is another **indeterminate form**. It could be one, because one to any power is one. Or it could be infinity, because it began as one and a tiny bit more, which grows large when raised to infinity.

If you encounter a limit that produces this form, you will need to transform the expression. The key is to raise the number *e* to the natural log of the expression. This equals the original expression.

A subliminal problem

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \to \infty(0)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \to \frac{0}{0}$$

$$\lim_{x\to\infty} \frac{\left[\frac{1}{\left(1+\frac{1}{x}\right)}\right]\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

So this limit is equal to one. Now look back at the original problem.

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1} = \boxed{1}$$

Caution: This limit doesn't equal the original limit; it's a sub-problem.

Once you have transformed the original limit, you can focus on the expression to which *e* is raised.

This new limit does not equal the original limit. It is a sub-problem. It produces an indeterminate product, which you must transform into an indeterminate quotient.

Now you can use **L'Hôpital's rule**. The sub-problem limit equals one.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x}$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

Plug in the answer from the subliminal problem.

$$= e^{(1)}$$

$$= \boxed{e}$$

The answer is *e*.

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$$

By plugging in the value of the limit in the sub-problem, you can evaluate the original limit. Since *e* raised to the first power is still *e*, that's your answer.

Some mathematicians use this limit expression as an alternate definition for *e*.

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Another Example of One to the Infinite Power

key concepts:

- Some **indeterminate forms** have to be transformed before you can apply **L'Hôpital's rule**.
- In order to apply L'Hôpital's rule to a limit of the form 1^∞ use the properties of logarithms to rewrite the exponent as a logarithm.

Another example of 1^∞

Evaluate: $\lim_{x\to1} x^{\frac{1}{(1-x)}}$

$$\lim_{x\to1} x^{\frac{1}{(1-x)}} = \lim_{x\to1} e^{\ln\left(x^{1/(1-x)}\right)}$$

$$= \lim_{x\to1} e^{\left(\frac{1}{(1-x)}\right)\ln x}$$

Use Fact 2 to write the exponent as a coefficient.

Remember The Facts!

1	$e^{\ln(\text{anything})} = (\text{anything})$
2	$\ln\left(A^B\right) = B\ln A$

When you encounter the **indeterminate form** 1^∞, you will need to make use of two facts about exponents and logarithms.

The first is that *e* raised to the natural log of any expression is equal to that same expression.

The second is that when there is an exponent inside a natural log expression, it can be moved to the outside as a factor.

$$\lim_{x\to1} x^{\frac{1}{(1-x)}} = \lim_{x\to1} e^{\ln\left(x^{1/(1-x)}\right)}$$

$$= \lim_{x\to1} e^{\left(\frac{1}{(1-x)}\right)\ln x}$$

There is no equal sign. This limit does **not** equal the original limit.

On The Side

$$\lim_{x\to1} \frac{1}{(1-x)}\ln x = \lim_{x\to1} \frac{\ln x}{1-x}$$

$$= \lim_{x\to1} \frac{\left(\frac{1}{x}\right)}{-1}$$

$$= \lim_{x\to1} \frac{-1}{x} = \boxed{-1}$$

Now that you have rewritten the expression, you can evaluate an easier limit. Forget about *e* and take the limit of its exponent.

Remember that this sub-problem is not equal to the original limit. It is just a side calculation.

To evaluate the limit in the sub-problem, you will have to transform the expression to produce an indeterminate quotient. Then you can apply L'Hôpital's rule.

$$\lim_{x\to1} x^{\frac{1}{(1-x)}} = \lim_{x\to1} e^{\ln\left(x^{1/(1-x)}\right)}$$

$$= \lim_{x\to1} e^{\left(\frac{1}{(1-x)}\right)\ln x}$$

$$= \lim_{x\to1} e^{\boxed{-1}}$$

$$= \boxed{\frac{1}{e}}$$

On The Side

$$\lim_{x\to1} \frac{1}{(1-x)}\ln x = \lim_{x\to1} \frac{-1}{x} = \boxed{-1}$$

The limit from the sub-problem is equal to –1, but that is not the value of the original limit!

When you plug in the result of the side calculation, you get the value of the original limit.