Unit: Techniques of Integration **Module:** Trigonometric Substitution Strategy

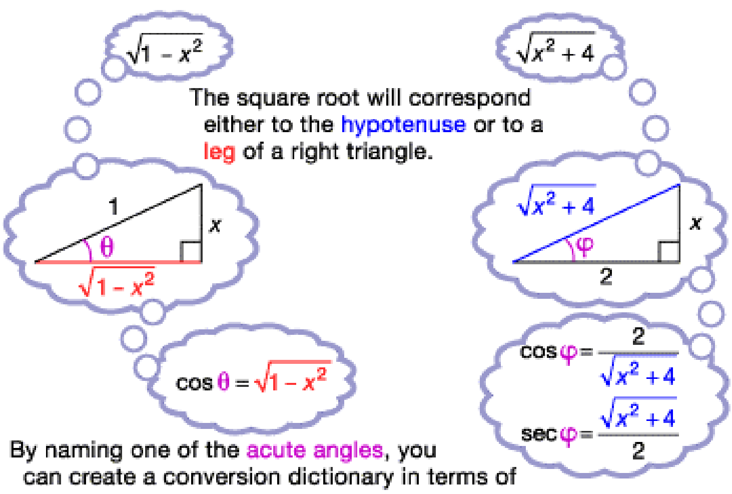
An Overview of Trig Sub Strategy

key concepts:

- Use **trigonometric substitution** to evaluate integrals involving the square root of the sum or difference of two squares.
  1. Match the square root expression with the sides of a right triangle.
  2. Substitute the corresponding trigonometric function into the integrand.
  3. Evaluate the resulting simpler integral.
  4. Convert from trigonometric functions back to the original variables.

Constructing a trig substitution triangle

Integrals involving the square root of the sum or difference of two squares are candidates for trigonometric substitution.



By naming one of the acute angles, you can create a conversion dictionary in terms of trigonometric functions, leading to the trigonometric substitution.

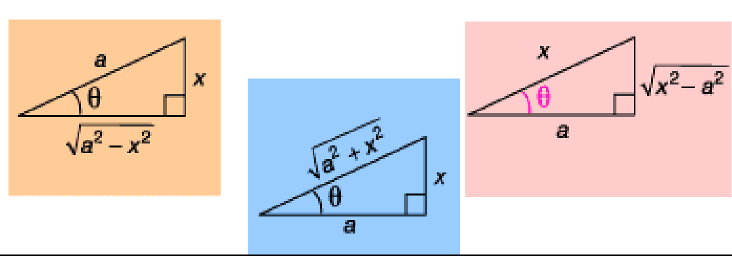
When you notice a radical expression or a rational power in an integrand, then the integral is a good candidate for **trigonometric substitution**.

The fact that the square root of the difference must correspond to the leg of the right triangle follows from the Pythagorean theorem. The square root of the sum of the squares is equal to the length of the hypotenuse, so to get the radical expression to fit the theorem you will have to match the other leg with the negative term underneath the radical. The same reasoning applies to why the square root of the sum corresponds to the hypotenuse.

Once you have generated your triangle, you can create a whole list of potential substitutions just by writing down the trigonometric expressions and finding what they are equal to in the triangle.

Trig substitution templates

If you see	TRIG-SUB Substitute	And use the identity
$a^2 - x^2$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$



The constant underneath the radical sign doesn't have to be one. It can be any number. You can find the square root of any positive number in the real number system.

You don't have to go back to the method of finding a trig substitution each time you evaluate one of these integrals. The table to the left illustrates exactly which substitutions you will find useful for each of the different possibilities.

Notice that there are three basic triangles you can create when you make a trig substitution. Each triangle has its own substitution that works best for it. Remember that you can switch the legs of a right triangle and not change it, so substitutions involving sine might be easier if you use cosine.

Calculus Lecture Notes

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Unit: Techniques of Integration **Module:** Trigonometric Substitution Strategy

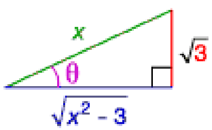
Trig Substitution Involving a Definite Integral – Part One

key concepts:

- Use **trigonometric substitution** to evaluate integrals involving the square root of the sum or difference of two squares.
  1. Match the square root expression with the sides of a right triangle.
  2. Substitute the corresponding trigonometric function into the integrand.
  3. Evaluate the resulting simpler integral.
  4. Convert from trigonometric functions back to the original variables.

Constructing a trig substitution triangle

Consider $\displaystyle\int_{\sqrt{3}}^{2}\frac{\sqrt{x^2-3}}{x}\,dx$.



! Notice that this is a definite integral, so you will have to evaluate the result at two and at $\sqrt{3}$ and calculate the difference.

✗ Since x is in the denominator, this integral does not lend itself to a u-substitution.

✓ However, the numerator is the difference of two squares under a radical, so you should try trig substitution.

$$?^2+3=x^2$$

$$?^2=x^2-3$$

$$?=\sqrt{x^2-3}$$

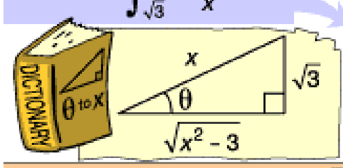
Keep only the positive root.

Trigonometric substitution can be used on definite integrals too. The same circumstances should be in place to use trigonometric substitution on a definite integral. There should be a radical made up of the difference or sum of squares and the integral should not lend itself to a simple u-substitution.

Substituting into a new integral

Consider $\displaystyle\int_{\sqrt{3}}^{2}\frac{\sqrt{x^2-3}}{x}\,dx$.

! Ignore the endpoints for the moment.



$$\sin\theta=\frac{\sqrt{3}}{x} \qquad \cos\theta=\frac{\sqrt{x^2-3}}{x}$$

$$\frac{d}{dx}(\sin\theta)=\frac{d}{dx}\left(\frac{\sqrt{3}}{x}\right)$$

$$\cos\theta\frac{d\theta}{dx}=-\frac{\sqrt{3}}{x^2}$$

$$\cos\theta\,d\theta=-\frac{\sqrt{3}}{x^2}dx$$

$$dx=-\frac{x^2}{\sqrt{3}}\cos\theta\,d\theta$$

$$x=\frac{\sqrt{3}}{\sin\theta}$$

$$x^2=\frac{3}{\sin^2\theta}$$

$$\int\frac{\sqrt{x^2-3}}{x}dx=\int\cos\theta\,dx$$

$$=-\frac{1}{\sqrt{3}}\int\cos^2\theta\,x^2\,d\theta$$

$$=-\frac{1}{\sqrt{3}}\int\cos^2\theta\frac{3}{\sin^2\theta}d\theta$$

$$=-\frac{3}{\sqrt{3}}\int\cot^2\theta\,d\theta$$

Use the form of the Pythagorean theorem involving cotangent.

$$=-\frac{3}{\sqrt{3}}\int(\csc^2\theta-1)\,d\theta$$

$$=\boxed{-\frac{3}{\sqrt{3}}(-\cot\theta-\theta)+C}$$

There are two ways to work a definite integral. One way is to convert the limits of integration when you make your trigonometric substitution. It is recommended that you avoid that process however, since there are lots of places where you might make a mistake. The better option is to ignore the limits of integration and work the problem like an indefinite integral first.

Here the integral has a radical made up of the difference of squares. So set it equal to one of the legs of the triangle. Since the x-term is positive, it must represent the hypotenuse.

Notice that when you take the derivative of the cosine term to find dx you get a result with x-terms in it. Don't let that confuse you. Just substitute out those x-terms with another trigonometric substitution.

Unit: Techniques of Integration **Module:** Trigonometric Substitution Strategy

Trig Substitution Involving a Definite Integral – Part Two

key concepts:

- When computing a definite integral using substitution, first ignore the limits of integration and treat the integral like an indefinite integral. Convert back to the original variable before evaluating at the endpoints.
- The labeled right triangle serves as a dictionary for making trig substitutions.

Continue evaluating $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

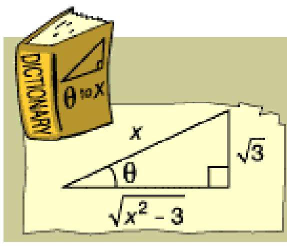
$$\int \frac{\sqrt{x^2-3}}{x}\,dx = -\frac{3}{\sqrt{3}}(-\cot\theta-\theta)+C$$

$$=\frac{3}{\sqrt{3}}(\cot\theta+\theta)+C$$

Factor out negative one and cancel.

$$=\frac{3}{\sqrt{3}}\left[\frac{\sqrt{x^2-3}}{\sqrt{3}}+\arcsin\left(\frac{\sqrt{3}}{x}\right)\right]+C$$

Substitute from the dictionary.



$$\cot\theta=\frac{\sqrt{x^2-3}}{\sqrt{3}} \qquad \sin\theta=\frac{\sqrt{3}}{x}$$

$$\theta=\arcsin\left(\frac{\sqrt{3}}{x}\right)$$

Use the inverse sine function to solve for θ.

In the previous lecture you used a trigonometric substitution to evaluate the indefinite integral corresponding to this definite integral. Now it is time to express the answer in terms of *x*.

Notice that to find θ in terms of *x* you will need to use an inverse trigonometric function. You can use any of the different inverse trig functions, but inverse sine is used here since it is the most frequently seen.

Returning to the definite integral

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \frac{3}{\sqrt{3}}\left[\frac{\sqrt{x^2-3}}{\sqrt{3}}+\arcsin\left(\frac{\sqrt{3}}{x}\right)\right]_{\sqrt{3}}^{2}$$

Convert to a definite integral.

$$=\frac{3}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}+\frac{\pi}{3}\right)-\frac{3}{\sqrt{3}}\left(0+\frac{\pi}{2}\right)$$

$$=\frac{3}{3}+\frac{3}{\sqrt{3}}\cdot\frac{\pi}{3}-\frac{3\pi}{2\sqrt{3}}$$

Distribute.

$$=1+\frac{\pi}{\sqrt{3}}-\frac{3\pi}{2\sqrt{3}}$$

Simplify.

$$=1+\frac{2\pi}{2\sqrt{3}}-\frac{3\pi}{2\sqrt{3}}$$

Find a common denominator.

$$=1-\frac{\pi}{2\sqrt{3}}$$

$$=1-\frac{\pi}{2\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

Multiply to remove the square root from the denominator.

$$=\boxed{1-\frac{\sqrt{3}\pi}{6}}$$

Now that you have found the solution of the indefinite integral, you are ready to evaluate the definite integral at the limits of integration.

After much algebra, the evaluation of the definite integral is complete.