Calculus Lecture Notes

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Unit: Elementary Functions and Their Inverses **Module:** Calculus of Inverse Functions

Derivatives of Inverse Functions

key concepts:

- You can calculate the derivative of an **inverse function** at a point without determining the actual inverse function.

Given that f is an invertible function:

- If f is continuous then f^{-1} is continuous.
- If f is differentiable then f^{-1} is differentiable.
- If f is increasing then f^{-1} is increasing.
- If f is decreasing then f^{-1} decreasing.

The **inverse** of a function retains many of the properties of the original function.

The derivative of the inverse

Given an invertible function $f(x)$, find $\frac{d}{dx}\left[f^{-1}(x)\right]$.

$f(f^{-1}(x)) = x$ Start with what you know.

$\frac{d}{dx}\left[f(f^{-1}(x))\right] = \frac{d}{dx}[x]$ Take the derivative of both sides. Use the chain rule.

$f'(f^{-1}(x)) \cdot \frac{d}{dx}\left[f^{-1}(x)\right] = 1$

$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$ Dividing by $f'(f^{-1}(x))$ isolates the derivative of the inverse.

To derive the formula for the derivative of an inverse, start with a relationship you know.

The composition of a function and its inverse is equal to x. You need to use the chain rule to differentiate both sides of that relationship.

Isolate the derivative of the inverse by dividing.

So if you know the value of the inverse at a point you can find the derivative of the inverse at that point.

Example!

Given $f(x) = 2x + \cos x$ and $f^{-1}(\pi) = \frac{\pi}{2}$, find $\frac{d}{dx}\left[f^{-1}(\pi)\right]$.

$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$ Start by using the formula derived earlier.

$= \frac{1}{2 - \sin\left[f^{-1}(x)\right]}$ Plug in the derivative.

$\frac{d}{dx}\left[f^{-1}(\pi)\right] = \frac{1}{2 - \sin\left[f^{-1}(\pi)\right]}$ Plug in the x-value.

$= \frac{1}{2 - \sin(\pi/2)}$ Use the fact that $f^{-1}(\pi) = \frac{\pi}{2}$.

$= \frac{1}{2-1} = \boxed{1}$ Evaluate the expression to get the derivative of the inverse at π.

In this example, you know the function and the value of the inverse at π. Your mission is to find the value of the derivative of the inverse at π.

Use the formula that you learned above. The derivative of $f(x) = 2x + \cos x$ is $f'(x) = 2 - \sin x$. Sine takes on values between –1 and 1, so the derivative lies between 1 and 3. It's always positive, which means the function is increasing. Remember that increasing functions are invertible.

Once you have found the derivative of the original function and verified that the function is invertible, all you have to do is plug into the formula.

You have evaluated the derivative of the inverse of a function at a point, without determining the inverse itself!