Calculus Lecture Notes

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Unit: Sequences and Series **Module:** Convergence and Divergence

Properties of Convergent Series

key concepts:

- The terms of separate convergent series can combine to form new convergent series. Also, constant multiples can be factored into and out of convergent series.
- If a series converges, then the terms must be shrinking to zero.

Adding and subtracting infinite series

Consider two convergent series.

$$\sum_{n=1}^{\infty} a_n = A \qquad \sum_{n=1}^{\infty} b_n = B$$

Consider the series composed of the sum of these two series.

sum $$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$ sum

The series converges to $A+B$.

The series composed of the difference of two convergent series also converges.

difference $$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$ difference

The series formed by multiplying each term of a convergent series by a constant c converges.

multiplying by a constant c $$\sum_{n=1}^{\infty} c a_n = cA$$ multiplying by a constant c

Convergent series follow many of the same rules of algebra that limits follow.

For example, if you add the terms of two series together the resulting sum is equal to the sum of the two series.

This works because you can break the series into two different series in the same way you can break a limit up. All of these rules parallel the **limit laws**.

The different of the terms of two series is equal to the difference of their respective sums.

Finally, if you have a convergent series and you multiply its terms by some constant, the resulting series is equal to the sum of the original series times that constant.

Classic mistake

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

! However, the opposite of this is *incorrect*. !

NO!	You cannot say	NO!

WRONG WRONG

If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Instead, if $\lim_{n\to\infty} a_n = 0$, then the associated series might converge *or* might not. You CANNOT conclude anything.

One of the conditions for a series to converge is that the terms must approach zero. Otherwise the sum of an infinite number of terms would have to be infinite.

But watch out! It is a classic mistake to assume that any series whose terms approach zero must converge.

Some series have terms that approach zero, but they don't approach zero fast enough. As a consequence, the entire series diverges.

So if your series converges, you know the limit of the sequence of terms is equal to zero. Is it possible to use that fact in another way?

Unit: Sequences and Series **Module:** Convergence and Divergence

The n th Term Test for Divergence

key concepts:

- You can use the ***n*****th-term test** for divergence to quickly tell if a series might converge:

  Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ must diverge.

Testing a series for convergence or divergence

quicky test

Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Q? Why is this true?

Suppose $\lim_{n\to\infty} a_n \neq 0$, and yet $\sum_{n=1}^{\infty} a_n$ converges.

Remember . . . If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$, which is not the original assumption.

A: Since the limit of the nth-term is not zero, the series can not converge, it must diverge.

Most of the questions you are asked when dealing with series relate to whether or not the series converges. As it turns out, there is a quick test you can use to identity many divergent series. This test is called the ***n*****th-term test**.

The nth-term test (also called the "quicky test") states that if the limit of the sequence of terms of the series does not approach zero, then the series must diverge.

Remember, you already know that if a series converges then the terms must approach zero. Consider a case where a series did converge but the limit didn't approach zero. This assumption creates a contradiction, so you know it is false.

The "quicky test" does not detect all divergent series, only those whose terms do not approach zero. It is possible for the terms to approach zero but for the series to still diverge.

Using the quicky test (the nth-term test)

Example!

Consider $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$.

The quicky test: Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Use the quicky test.

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{2n}{2n} \quad \text{Use L'Hôpital's rule.}$$

$$= 1$$

$$\neq 0$$

Since the limit of the terms is not zero, the series diverges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

Use the quicky test.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Since the limit of the terms is zero, you do not know if the series converges.

You need other techniques to determine if this series converges or not.

Consider this example.

You can use the "quicky test" to check for convergence.

Notice that this limit is an indeterminate form. Use L'Hôpital's rule to evaluate the limit.

Since the terms are not approaching zero, the series **must** diverge.

In this example, the terms are approaching zero.

The "quicky test" is inconclusive when the terms approach zero. The series might converge; it might not.

You will need other techniques to determine if this series converges.