Calculus Lecture Notes

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Unit: Applications of Integration **Module:** Shells

Introducing the Shell Method

key concepts:

- The volume of a **solid of revolution** using the **shell method** where x is the radius and $h(x)$ is the height of an arbitrary shell is V where:

$$V = \int_a^b 2\pi x\, h(x)\, dx\,.$$

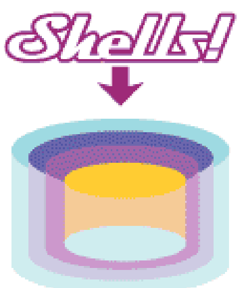
Coring the racetrack

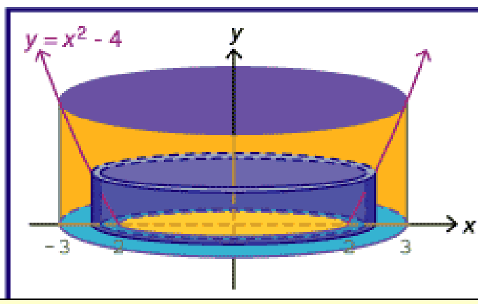
Consider the region bounded by:
- the graph of $y = x^2 - 4$
- the x-axis
- the graph of $x = 2$
- the graph of $x = 3$

Q: Is there a different method which gives the same answer?

A: Try coring the solid.

As you core further out, the shells become wider and taller.
As you core closer in, the shells become narrower and shorter.





You have already found the volume of this **solid of revolution** using the **washer method**. But is there another way of determining the volume?

Instead of slicing the volume into washers, what happens if you core the solid instead? Think of cutting little cylinders out of the solid

Notice that the solid can be described by an infinite number of cylinders whose radius and height are given by the equations bounding the rotated region.

The method of finding volume using these cylinders is called the **shell method**.

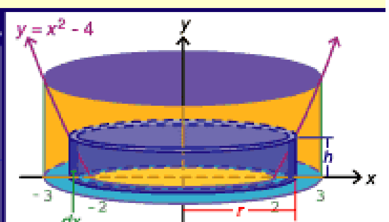
Integrating the cores

Find the volume of the racetrack.

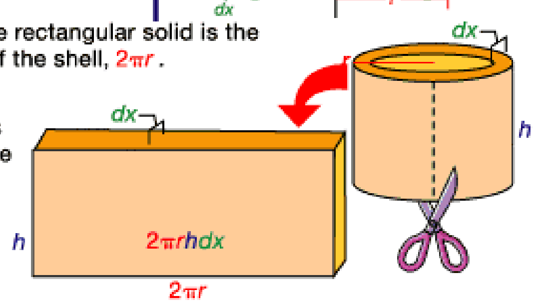
$$V = \int_2^3 2\pi x(x^2-4)\,dx$$

$$= 2\pi\int_2^3 (x^3-4x)\,dx$$

Factor out 2π and distribute x.



- The length of the rectangular solid is the circumference of the shell, $2\pi r$.
- The height is h.
- The thickness is dx, a tiny change in x.



$$= 2\pi\left[\frac{x^4}{4} - 2x^2\right]_2^3$$

$$= 2\pi\left[\left(\frac{81}{4} - 18\right) - \left(\frac{16}{4} - 8\right)\right] = \frac{25}{2}\pi$$

The shell method gives the same answer as the washer method.

Consider the volume of a thin right cylindrical shell. The shell itself can be cut open. When you do this, you are left with a rectangular solid with a very tiny thickness.

The volume of shell is equal to the circumference of the circle times the height times the thickness. The thickness is a small change in x.

The circumference is equal to $2\pi r$, where r is equal to the distance from the axis of revolution. Here $r = x$.

The height of the cylinder is equal to the distance between the parabola and the x-axis. This distance is given by $x^2 - 4$.

Multiplying all those pieces together gives you the volume of an arbitrary cylindrical shell.

To find the volume, just integrate. Notice that the limits of integration run along the x-axis since that is where the shells stack.

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Why Shells Can Be Better Than Washers

key concepts:

- The volume of a **solid of revolution** using the **shell method** where x is the radius and $h(x)$ is the height of an arbitrary shell is V where:

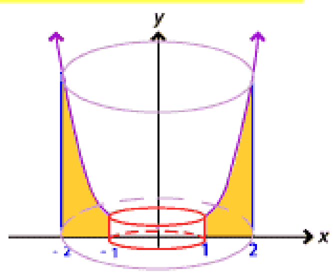
$$V = \int_a^b 2\pi x\, h(x)\, dx\,.$$

- For some solids, the shell method of finding volume is simpler than the **washer method**.

An application of the washer method

Consider the region bounded by
- the graph of $y = x^4 + 1$
- the graph of $x = 1$
- the graph of $x = 2$
- the x-axis

Revolve the region about the y-axis.



Q: Why is it not a good idea to use washers for this solid?

A: The washers change shape, which makes the integration more difficult. You would need two integrals.

Consider the **solid of revolution** described to the left. Start by graphing it.

Notice that if you use the **washer method** to evaluate the volume the washers would stack vertically. The integral would be expressed in terms of y which requires you to do some algebra.

There is another reason not to use the washer method. Notice that the washers near the bottom of the solid are defined differently than the washers at the top. If you used washers to find the volume, you would have to evaluate two integrals.

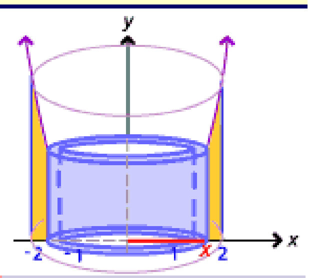
Try using the **shell method** instead.

Avoiding the change in shape

Q: What is the volume of a shell?

A: The volume of a shell is base • height • width.

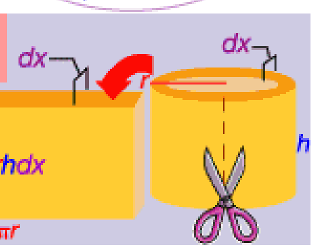
$$V = \int_1^2 2\pi x\,(x^4 + 1)\, dx$$



The base of the rectangular solid is the circumference of the shell, $2\pi x$.

The height is $(x^4 + 1)$.

The thickness is dx.



$$V = 2\pi\left(\frac{x^6}{6} + \frac{x^2}{2}\right)\Bigg|_1^2 = 2\pi\left[\left(\frac{76}{6} - \frac{4}{6}\right)\right] = 2\pi\left(\frac{72}{6}\right) = 24\pi$$

By choosing shells over the washers, you avoid having to evaluate two integrals.

To use the shell method, start by drawing an arbitrary shell.

Remember that shells are simply curled up rectangles. By uncurling the shell you can find a simple way of describing its volume.

Notice that the radius of the shell is x. Use the radius to find the circumference of the shell. The circumference of the shell is equal to the base of the rectangular solid.

The height of the shell is equal to the height of the rectangle. In this example the height can be found by using the equation of the curve.

The thickness of the shell is a small change in x. The shells range from $x = 1$ to $x = 2$.

Now you have all the pieces you need to find the volume. The actual integral is the easy step here.

You avoid having to break the integral into two pieces by using the shell method.

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The Shell Method – Integrating with Respect to y

key concepts:

- The volume of a solid of revolution using the **shell method** where y is the radius and $h(y)$ is the height of an arbitrary shell is V where:

$$V = \int_a^b 2\pi y\, h(y)\, dy\,.$$

A horizontal region

Consider the region bounded by

- the graph of $y = \sqrt{x}$
- the y-axis
- the graph of $y = 1$
- the graph of $y = 4$

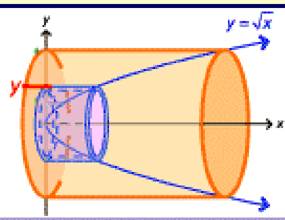
Revolve the region about the x-axis.

Consider the solid of revolution described to the left. To find the volume, it is a good idea to start by graphing the solid.

Notice that the region is revolved around the x-axis. Always make sure that you revolve the region about the correct line. Using the wrong axis of revolution will greatly change your graph.

Using shells with respect to y

If you use washers, their shape will change part of the way through. You will need two separate integrals.

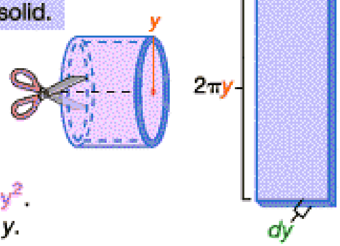


Q: What is the volume of each cored piece?

A: The volume of a shell is $2\pi y(y^2)\,dy$.

Each piece unfolds into a rectangular solid.

- The radius is y.
- The length is the circumference of the shell, $2\pi r$ or in this case $2\pi y$.
- The width is the x-value associated with a given y-value.
- $y = \sqrt{x}$, so $x = y^2$, and the width is y^2.
- The thickness is dy, a tiny change in y.



Consider using the **washer method** on the solid of revolution first. From $x = 0$ to $x = 1$, the washer is defined by two lines. From $x = 1$ to $x = 16$ the washer is defined by a line and a parabola. The way the washer is defined changes. Washers might not be the easiest way to find the volume.

Now consider the **shell method**. Shells could be stacked around each other starting at $y = 1$ and running to $y = 4$. More importantly, the shells would always be defined by the y-axis and the parabola.

Looking at an arbitrary shell, you can see that the radius of the shell is equal to y. Use the radius to find the circumference of the shell.

Notice that the width of the shell is equal to x. The width must be expressed in terms of y since the thickness is a small change in y.

Multiply all the pieces together and you are ready to integrate.

$$V = \int_1^4 2\pi y(y^2)\,dy$$
$$= \int_1^4 2\pi y^3\,dy$$
$$= 2\pi \int_1^4 y^3\,dy$$
$$= 2\pi\left(\frac{y^4}{4}\right)\Bigg|_1^4$$
$$= 128\pi - \frac{\pi}{2}$$
$$= \frac{256\pi - \pi}{2}$$
$$= \boxed{\frac{255\pi}{2}}$$

By translating your sketch into an integral, you arrive at the volume of the solid.

Since the shells stack from 1 to 4, those are the limits of integration. Multiply the circumference, height, and thickness together to express the volume of an arbitrary shell.

Notice that the integration is pretty straightforward. Setting up the integral is the tricky part.