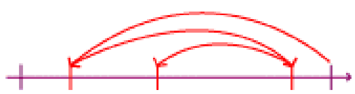
# Absolute and Conditional Convergence

**key concepts:**

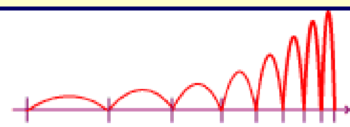
- If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ **converges absolutely**.
- A series is **conditionally convergent** if it converges yet does not converge absolutely. Absolute convergence is a stronger type of convergence than conditional convergence.

## Absolute convergence



$\sum_{n=1}^{\infty} a_n$ converges

Does it converge because the terms alternate between positive and negative?



Consider $\sum_{n=1}^{\infty} |a_n|$

Does this series converge?

Note: $\sum_{n=1}^{\infty} |a_n|$ may converge to a different number than $\sum_{n=1}^{\infty} a_n$.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

When you studied series whose terms were all positive, there was only one kind of convergence.

If a series has a mixture of positive terms and negative terms, they cancel each other out to a certain extent. It can be interesting to make the terms all positive and see if they still converge. If they do, then the series is said to be **absolutely convergent**.

## Testing for absolute convergence

**Example!** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

This is an alternating series. Apply the alternating series test:

1. $\frac{1}{n^2} > 0$
2. $\frac{1}{1^2} > \frac{1}{2^2} > \frac{1}{3^2} > \cdots$
3. $\lim_{n\to\infty} \frac{1}{n^2} = 0$

It converges.

This series meets the conditions of the **alternating series test**. The $a_n$ values are positive, decreasing and approaching zero.

**Q:** Does it converge absolutely?

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

It is a p-series with $n>1$, and it converges.

**A:** Yes, it converges absolutely!

To test if the series converges absolutely, you must take the absolute value of the terms. In this case, the result is a $p$-series with $n > 1$, which converges. Therefore the series converges absolutely.

# Absolute and Conditional Convergence

## Conditional convergence

**Example!** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

This is an alternating series. Apply the alternating series test:

1. $\frac{1}{n} > 0$
2. $\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \cdots$
3. $\lim_{n\to\infty} \frac{1}{n} = 0$

It converges.

**Q:** Does it converge absolutely?

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ The harmonic series, which diverges.

**A:** No, it does not converge absolutely.

Consider this **alternating series**, whose denominator is just *n*.

This series also converges by the alternating series test.

However, when you take the absolute value of the terms it produces the harmonic series, which diverges.

Since this series converges, but not absolutely, you can say it **converges conditionally**.

**Example!** $\sum_{n\to 1}^{\infty} \frac{\cos n}{n^3}$

**Q:** Does it converge absolutely?

Look at the absolute value of the term.

Consider $\sum_{n\to 1}^{\infty} \left|\frac{\cos n}{n^3}\right|$

Make a comparison: $\left|\frac{\cos n}{n^3}\right| \le \frac{1}{n^3}$ $\quad |\cos n| \le 1$

$\sum_{n\to 1}^{\infty} \frac{1}{n^3}$ converges because it is a *p*-series with $n>1$.

**A:** Yes, it converges absolutely.

This series has positive and negative terms, but they are not alternating. You cannot use the alternating series test.

If you study the absolute value of the terms, you can compare it to the *p*-series with $p = 3$, which converges. Since $\cos n$ is less than or equal to one, the first series is dominated by the convergent *p*-series, and therefore converges by the **direct comparison test**.

Thus the example converges absolutely. That means that it converges in its original form, too.

## Conditional vs. absolute convergence

| Absolute Convergence | | Conditional Convergence |
|---|---|---|
| Very strong: if absolute value is applied to the terms in series, it still converges. | Strength | Weaker: if absolute value is applied to the terms in the series, it will diverge. |
| Not important: if the terms in the series are reordered, the series always converges to the same number. | Order of Terms | Very important: the terms in the series may be reordered to sum **any** value. |

Absolute convergence is stronger than conditional convergence. If a series converges absolutely, then it converges in its own right and it converges when you take the absolute value of its terms.

Conditional convergence means that the series only converges because it has positive and negative terms that offset each other. If they were all positive, they would sum to infinity.

If a series converges absolutely, the order of the terms does not matter. You will always get the same result, like with finite addition. If a series converges conditionally, then infinite addition gets weird. If you reorder the terms you get a different result, and you can get find a way to get any result you want.