Unit: Applications of Integration **Module:** Moments and Centers of Mass

Center of Mass

key concepts:

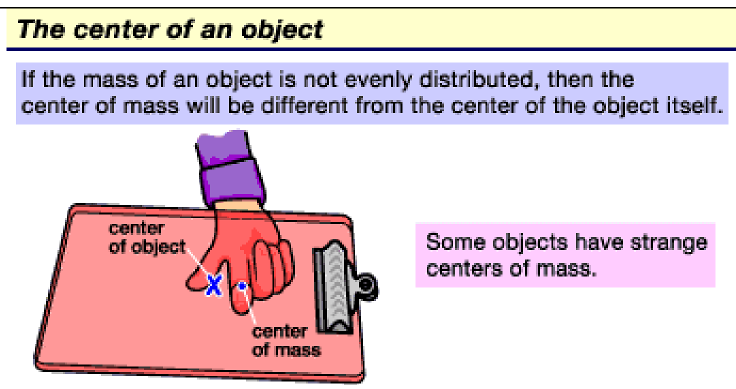
- The **center of mass** of an object is the point where you can assume all the mass is concentrated.
- A **moment** can be thought of as a system's tendency to rotate about an axis.

The **moment about the *y*-axis** is $\sum_{n=1}^{N} m_n x_n$.

The **moment about the *x*-axis** is $\sum_{n=1}^{N} m_n y_n$

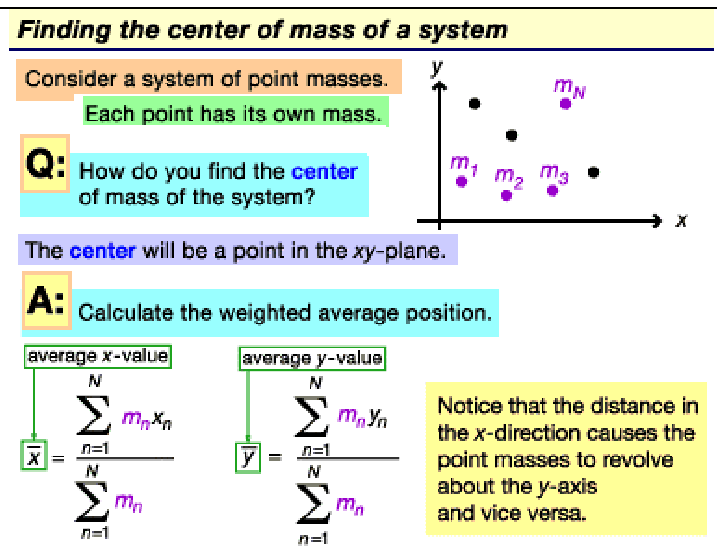
- The center of mass, $(\bar{x}, \bar{y})$, can be found by dividing the moments by the total mass:

$$\bar{x} = \frac{\sum_{n=1}^{N} m_n x_n}{\sum_{n=1}^{N} m_n}, \quad \bar{y} = \frac{\sum_{n=1}^{N} m_n y_n}{\sum_{n=1}^{N} m_n}.$$



The **center of mass** is the point where you can assume all the mass is concentrated. If you support an object at the center of mass, the object will balance.

When dealing with objects of uniform mass, the center of mass is located at the center of the object. But not all objects have uniform mass. The center of mass in these objects might seem strange.



Consider a system of point masses. A point mass is a point where you assume mass is concentrated.

What is the center of mass of the system?

Since all the mass is on the flat plane, then the center of mass will be on the plane too.

Here are the formulas for finding the coordinates of the center of mass. The numerator of the *x*-coordinate is also known as the **moment about the *y*-axis**. The numerator of the *y*-coordinate is known as the **moment about the *x*-axis**. A **moment** can be thought of as the tendency for an object to rotate.

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Center of Mass of a Thin Plate

key concepts:

- The **center of mass** of an object is the point where you can assume all the mass is concentrated.
- The center of mass of a thin plate (planar lamina) of uniform density is located at $(\bar{x}, \bar{y})$ where:

$$\bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\,dx \text{ and } \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx.$$

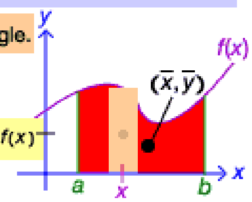
The center of mass of a plate defined by a function

Q: Where is the center of mass of a thin plate described by the curve of $f(x)$?

First, find the x-center. Insert an arbitrary rectangle.

The location of the rectangle weights its area.

Find an expression for the weighted value of the rectangle, given by area times location.



$$\bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\,dx$$

A: The center of mass is located at the point $(\bar{x}, \bar{y})$.

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx$$

Finding the **center of mass** of a continuous region or a thin plate is different from a system of point masses because there are an infinite number of points to consider. But you can find the center of mass using a definite integral.

Start by considering an arbitrary rectangle in the region. The rectangle has a very thin base and its height is defined by the curve of f.

The height times the width tells you the area of the rectangle. Multiplying by the distance x weights it, since that value is the center of the rectangle. Integrating and dividing by the area gives you the x-coordinate of the center.

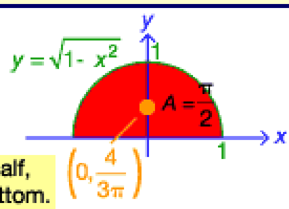
A similar process works for the y-coordinate. The height times the width tells you the area. Half the height, or $f(x)/2$, weights the rectangle.

Center of mass of a semicircle

Example Consider a semicircle.

Since the semicircle is symmetric about the y-axis, the x-coordinate of the center of mass will be zero.

The y-coordinate will be less than one-half, because most of the area is near the bottom.



$$\bar{y} = \frac{\cancel{2}}{\pi}\int_{-1}^{1} \frac{1}{\cancel{2}}(1-x^2)\,dx = \frac{1}{\pi}\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]$$

$$= \frac{1}{\pi}\int_{-1}^{1}(1-x^2)\,dx = \frac{1}{\pi}\left(1-\frac{1}{3}+1-\frac{1}{3}\right)$$

$$= \frac{1}{\pi}\left(x-\frac{x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{1}{\pi}\left(2-\frac{2}{3}\right) = \frac{1}{\pi}\left(\frac{6}{3}-\frac{2}{3}\right) = \frac{1}{\pi}\left(\frac{4}{3}\right) = \boxed{\frac{4}{3\pi}}$$

The coordinates of the center of mass are $\left(0, \frac{4}{3\pi}\right)$ or $(0, 0.4244)$.

Consider a semicircle of radius one. Where is the center of mass?

The x-coordinate is easy to find. By symmetry, half of the mass is to the left of the y-axis and the other half is to the right. So the x-coordinate must be at the axis, or at $x = 0$.

You know that the y-coordinate will be lower than half the height, since there is more mass towards the bottom of the figure than the top.

Start by finding the area. The area of a semicircle is half the area of a circle.

Now use the formula and plug in the equation of a semicircle.

Evaluate the integral to find the y-coordinate.

Put the two pieces together and you have the location of the center of mass.