Unit: The Basics of Integration **Module:** Illustrating Integration by Substitution

Integrating Composite Trigonometric Functions by Substitution

key concepts:

- Integration by substitution is a technique for finding the antiderivative of a **composite function**. A composite function is a function that results from first applying one function, then another.
- If the *du*-expression is only off by a constant multiple, you can still use integration by substitution by moving that constant out of the integral.

Substituting arguments for u

Evaluate $\int (2x+1)\sin(x^2+x)\,dx$.

Let $u = x^2 + x$.
Then $du = (2x+1)dx$.

$$\int (2x+1)\sin(x^2+x)\,dx = \int (2x+1)\sin u\,dx$$ Substitute u.

$$= \int \sin u\,du$$ Express everything else in terms of u and du.

$$= -\cos u + C$$

$$\boxed{= -\cos(x^2+x) + C}$$ Leave the answer in terms of x.

This integral involves a **composite function**: the sine of a complicated expression. If you let u be the inside of the function, notice that du is found surrounding the sine function.

After you substitute u, make sure that there are nothing remains in terms of x.

Recall that the derivative of $\sin x$ is $-\cos x + C$.

Make sure to replace u with its expression in terms of x.

CHECK YOUR WORK

$$\frac{d}{dx}[-\cos(x^2+x)+C] = -\frac{d}{dx}[\cos(x^2+x)] + \frac{d}{dx}[C]$$
$$= -[-\sin(x^2+x)]\frac{d}{dx}(x^2+x) + 0$$
$$= (2x+1)\sin(x^2+x)$$

You can check that your answer is correct by taking its derivative.

Evaluate $\int \cos(4x)\,dx$.

Let $u = 4x$.
Then $du = 4\,dx$.
So $\frac{1}{4}du = dx$.

$$\int \cos(4x)\,dx = \int \cos u\,dx$$ Substitute u.

$$= \int \cos u \cdot \frac{1}{4}du$$ Substitute the du-expression.

$$= \frac{1}{4}\int \cos u\,du$$ **constant multiple rule**

$$= \frac{1}{4}\sin u + C$$

$$\boxed{= \frac{1}{4}\sin(4x) + C}$$ Leave the answer in terms of x.

Here is another composite function. Let u be the inside expression. When you find du, you will notice that there is no multiple of 4 in the integrand, just dx.

Since 4 is just a constant multiple, solve for dx and substitute that expression into the integrand.

You can move 1/4 outside the integrand since it is a constant multiple.

After you integrate, make sure to replace u with its expression in terms of x.

CHECK YOUR WORK

$$\frac{d}{dx}\left[\frac{1}{4}\sin(4x)+C\right] = \frac{d}{dx}\left[\frac{1}{4}\sin(4x)\right] + \frac{d}{dx}[C]$$
$$= \frac{1}{4}\frac{d}{dx}[\sin(4x)] + 0$$
$$= \cos(4x)$$

Take the derivative of your answer to make sure it is correct.

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Integrating Composite Exponential and Rational Functions by Substitution

key concepts:

- Integration by substitution is a technique for finding the antiderivative of a composite function. A **composite function** is a function that results from first applying one function, then another.
- You may need to experiment with several choices for u when using integration by substitution. A good choice is one whose derivative is expressed elsewhere in the integrand.

Composite exponential functions

Evaluate $\int (10x^4 + 4x)e^{(x^5 + x^2 - 1)}\,dx$.

$$\int \boxed{(10x^4 + 4x)}e^{\boxed{(x^5 + x^2 - 1)}}\,dx = \int (10x^4 + 4x)e^u\,dx$$ Substitute u.

$u = ?$

$$= \int e^u \cdot 2\,du$$ Express everything else in terms of u and du.

$$= 2\int e^u\,du$$

$$= 2e^u + C$$

$$\boxed{= 2e^{(x^5 + x^2 - 1)} + C}$$ Leave the answer in terms of x.

Let $u = x^5 + x^2 - 1$.
Then $du = 5x^4 + 2x^2\,dx$.
$2du = 2(5x^4 + 2x^2)\,dx$
$= 10x^4 + 4x^2\,dx$

The first step when integrating by substitution is to identify the expression that you will replace with u. There will often be many candidates for u. A good strategy is to pick one and test it. In this case, differentiating the expression in the first box produces $40x^3 + 4$, but there is no other cubic in the integrand.

Choosing the expression in the second box and differentiating gives you an expression with a fourth power and a first power. The exact expression is not in the integrand. However, it can be multiplied by 2 to give the expression in the first box.

Once you have determined the expression for u, the integrand should be simple to evaluate. Remember to replace u with its expression in terms of x.

Substitution and the reciprocal function

Evaluate $\int \frac{2x}{x^2 + 1}\,dx$.

Let $u = (x^2 + 1)$.
Then $du = 2x\,dx$.

$$\int \frac{2x}{x^2 + 1}\,dx = \int \frac{2x}{u}\,dx$$ Substitute u.

$$= \int \frac{du}{u} = \int \frac{1}{u}\,du$$ Express everything else in terms of u and du.

$$= \ln|u| + C$$

RECALL

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$= \ln|x^2 + 1| + C$$

$$= \boxed{\ln(x^2 + 1) + C}$$ Leave the answer in terms of x.

In the case of a rational integrand, the best choice for u may be the denominator. In this example, du then appears in the numerator.

Replace the expressions in terms of x with the corresponding u- and du-expressions.

The integral of du/u is $\ln|u| + C$.

You have not finished the technique until you have your result expressed in terms of x.

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More Integrating Trigonometric Functions by Substitution

key concepts:

- You can apply integration by substitution to integrands involving trigonometric functions that are not composite functions.
- When working with integrands that include trigonometric expressions, it is sometimes necessary to rewrite those expressions using trig identities.

The product of two trig functions

Evaluate $\int \sin x \cos x \, dx$.

Let $u = \sin x$.
$du = \cos x \, dx$

$$\int \sin x \cos x \, dx = \int u \cos x \, dx \quad \text{Substitute } u.$$
$$= \int u \, du$$
$$= \frac{u^2}{2} + C$$
$$= \boxed{\frac{\sin^2 x}{2} + C}$$

Note the difference:
$\sin^2 x = (\sin x)(\sin x)$
$\sin x^2 = \sin(x \cdot x)$

Instead of a composite function, this integral involves the product of two trigonometric functions.

You could let u be $\sin x$, in which case du would be $\cos x$, or you could let u be $\cos x$ making du be $-\sin x$. You might want to choose $u = \sin x$ to avoid the negative sign.

Once you have determined the expression for u, the integrand should be simple to evaluate. Remember to replace u with its expression in terms of x.

CHECK YOUR WORK

$$\frac{d}{dx}\left[\frac{\sin^2 x}{2} + C\right] = \frac{2\sin x}{2}\cos x + 0$$
$$= \boxed{\sin x \cos x}$$

You can check your work by integrating with the help of the chain rule.

Looking for a substitution

Evaluate $\int \tan x \, dx$.

Let $u = \cos x$.
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$
$$= \int \frac{\sin x}{u} dx \quad \text{Substitute } u.$$
$$= \int \frac{-1}{u} du = -\int \frac{du}{u}$$
$$= -\ln|u| + C$$
$$= \boxed{-\ln|\cos x| + C}$$

RECALL
$$\tan x = \frac{\sin x}{\cos x}$$

You may often find it useful to express trigonometric integrands in terms of the sine and cosine functions.

Notice that you must choose $u = \cos x$, since it is in the denominator. That way the du-expression can replace the numerator and dx.

Factor out the -1 from the integrand.

The integral of du/u is $\ln|u| + C$.

Make sure to express your result in terms of x.

CHECK YOUR WORK

$$\frac{d}{dx}\left[-\ln|\cos x| + C\right] = -\frac{1}{\cos x}(-\sin x) + 0$$
$$= \frac{\sin x}{\cos x} = \boxed{\tan x}$$

Check that your answer is correct by integrating.

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Choosing Effective Function Decompositions

key concepts:

- Experiment with different choices for *u* when using integration by substitution. A good choice is one whose derivative is expressed elsewhere in the integrand.
- When working with integrands that include trigonometric expressions, it is sometimes necessary to rewrite those expressions using trig identities.

What is the best choice for *u*? $\int \frac{xe^{x^2}}{e^{x^2}+5}\,dx$ $u = x$ $u = xe^{x^2}$ $u = e^{x^2}$ $u = x^2$ $u = e^{x^2}+5$ Let $u = e^{x^2}+5$. $du = 2xe^{x^2}\,dx$ | $\frac{du}{2} = xe^{x^2}\,dx$	When applying integration by substitution to composite functions there may be several choices for *u*. In the case of a rational function, the best choice is often the denominator In this example, *du*/2 produces the expression in the numerator.
$\int \sec^3 x \tan x\,dx = \int \frac{\sin x}{(\cos x)^4}\,dx$ $u = \tan x$ $u = \sec x$ $u = \sec^3 x$ $u = \sin x$ $u = \cos x$ Let $u = \cos x$. | $du = -\sin x\,dx$ $-du = \sin x\,dx$	You may want to express trigonometric integrands in terms of sine and cosine before integrating. Since the denominator has $\cos x$ raised to a power, choose *u* to be $\cos x$. Then –*du* produces the expression in the numerator.