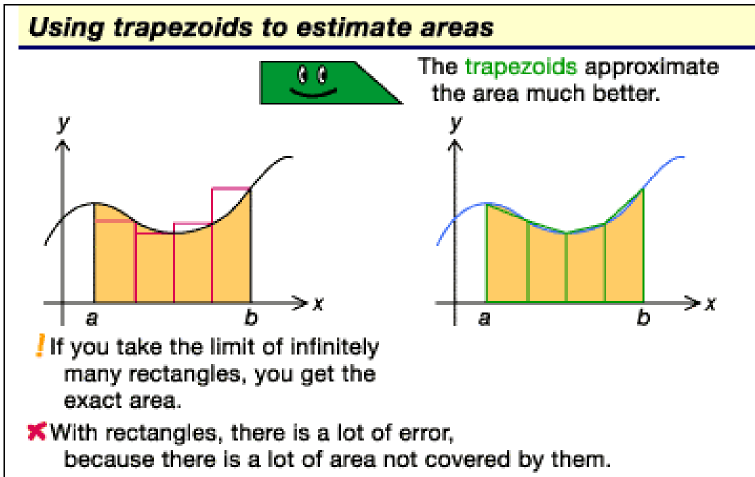
Unit: The Basics of Integration **Module:** Numerical Integration

Deriving the Trapezoidal Rule

key concepts:

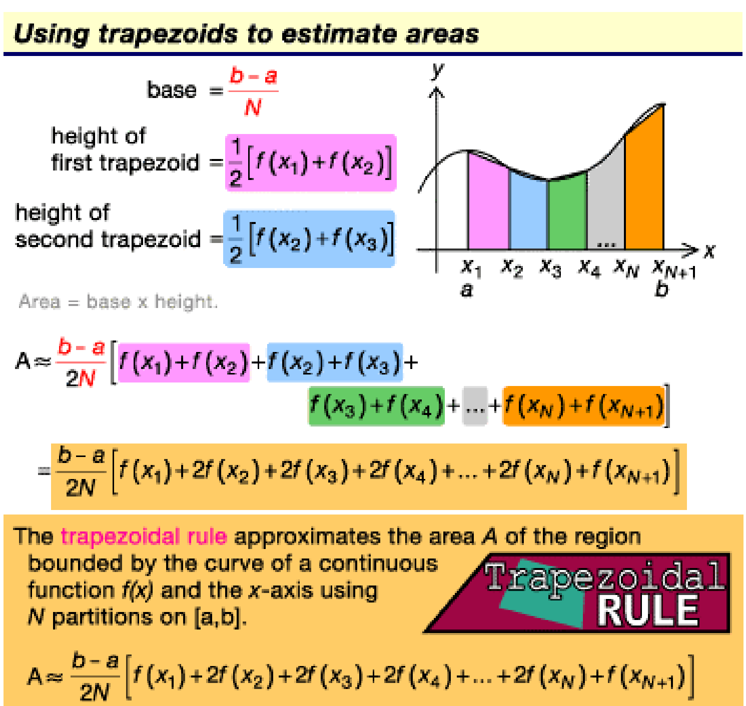
- The **trapezoidal rule** approximates the area A of the region bound by the curve of a continuous function $f(x)$ and the x-axis using N partitions on $[a,b]$.
- The trapezoidal rule: $A \approx \frac{b-a}{2N}\left[f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_N)+f(x_{N+1})\right]$.



When you take the integral to find the area under a curve, you are actually dividing the region into an infinite number of rectangles of arbitrarily small width and adding the areas of those rectangles together. But some functions can't be integrated. In these cases, you can approximate the definite integral and the area under the curve with a finite number of rectangles.

But notice that rectangles produce a lot of error.

However, trapezoids produce less error. This is because the trapezoid more closely emulates the partition that it is approximating.



To approximate an area using trapezoids, you first have to know how to find the area of a trapezoid. The area of a trapezoid is equal to the base times the average of its two heights.

Suppose you have a curve bounded on the interval $[a,b]$ and you need to approximate the area.

Start by deciding how many partitions you want to find. The more partitions, the more accurate your approximation. By the same token, more partitions also means more calculations. The number of partitions is called N.

Find the value of f for each x-value where you are breaking up your interval. Plug these values into this formula to approximate the area. This formula is called the **trapezoidal rule**.

There are a couple of pieces of the trapezoidal rule to watch out for. First, notice that you will have to evaluate N+1 terms if you use N partitions. Also notice that there are several twos floating around the formula. Don't leave any of those out!

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Example of the Trapezoidal Rule

key concepts:

- The **trapezoidal rule** approximates the area A of the region bound by the curve of a continuous function $f(x)$ and the x-axis using N partitions on $[a,b]$.
- The trapezoidal rule: $A \approx \frac{b-a}{2N}\left[f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_N)+f(x_{N+1})\right]$.

$$\int_1^3 \frac{1}{x}\,dx = \ln 3$$

How would you determine the numerical value for ln3?

$$\int_1^3 \frac{1}{x}\,dx = \ln|x|\,\Big|_1^3 = \ln 3 - \ln 1 = \ln 3$$

Use calculus!

Here's a strange situation. Suppose you are stranded on a desert island with a group of people and for some reason you need to know the value of the natural log of three? Hey, weirder things have happened.

Well, if you remember from Calculus that the integral of $1/x$ is equal to the natural log function, you can construct an integral whose area is equal to exactly ln 3.

How could you use this information to determine the value of ln 3?

The trapezoidal rule approximates the area A of the region bounded by the curve of a continuous function $f(x)$ and the x-axis using N partitions on [a,b].

Trapezoidal RULE

$$A \approx \frac{b-a}{2N}\left[f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+\ldots+2f(x_N)+f(x_{N+1})\right]$$

Since the only tools you have to compute numbers are what you know, then you will have to approximate the value of ln 3.

Here is the **trapezoidal rule**. If you use this rule your approximation will be better than those of the other people stranded on the island.

Applying the trapezoidal rule

$$\int_1^3 \frac{1}{x}\,dx = \ln 3 \approx \frac{67}{60}$$

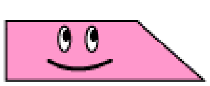
$$A \approx \frac{1}{2}\cdot\frac{1}{2}\left[1+2\left(\frac{2}{3}\right)+2\left(\frac{1}{2}\right)+2\left(\frac{2}{5}\right)+\frac{1}{3}\right]$$

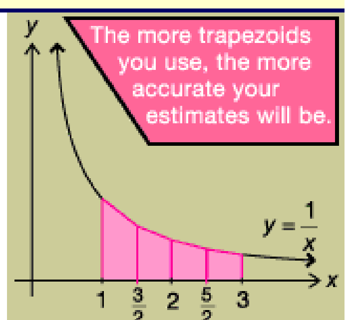
$$= \frac{1}{4}\left(1+\frac{4}{3}+1+\frac{4}{5}+\frac{1}{3}\right)$$

$$= \frac{1}{4}\left(\frac{15}{15}+\frac{20}{15}+\frac{15}{15}+\frac{12}{15}+\frac{5}{15}\right)$$

$$= \frac{1}{4}\left(\frac{67}{15}\right)$$

$$A \approx \frac{67}{60}$$





Check your work.

$\ln 3 \approx 1.098612289$

$\frac{67}{60} \approx 1.116666667$

The two values are very close.

Here the interval $[a,b]$ is divided into 4 regions. So $f(x) = 1/x$, $N = 4$, $a = 1$, and $b = 3$.

If you wanted an even more precise approximation, you could increase the number of rectangles used in the approximation.

Each partition has a base of 1/2. Start by finding what the x-values are at the partitions. Then find their corresponding $f(x)$-values. Plug these values into the trapezoidal rule.

Find a common denominator and add the fractions together.

Notice that your answer is very close to the actual value of ln 3.

Try doing this example with eight partitions to see if the approximation gets better. Also compare the trapezoidal rule to the rectangular approximation.