Calculus Lecture Notes

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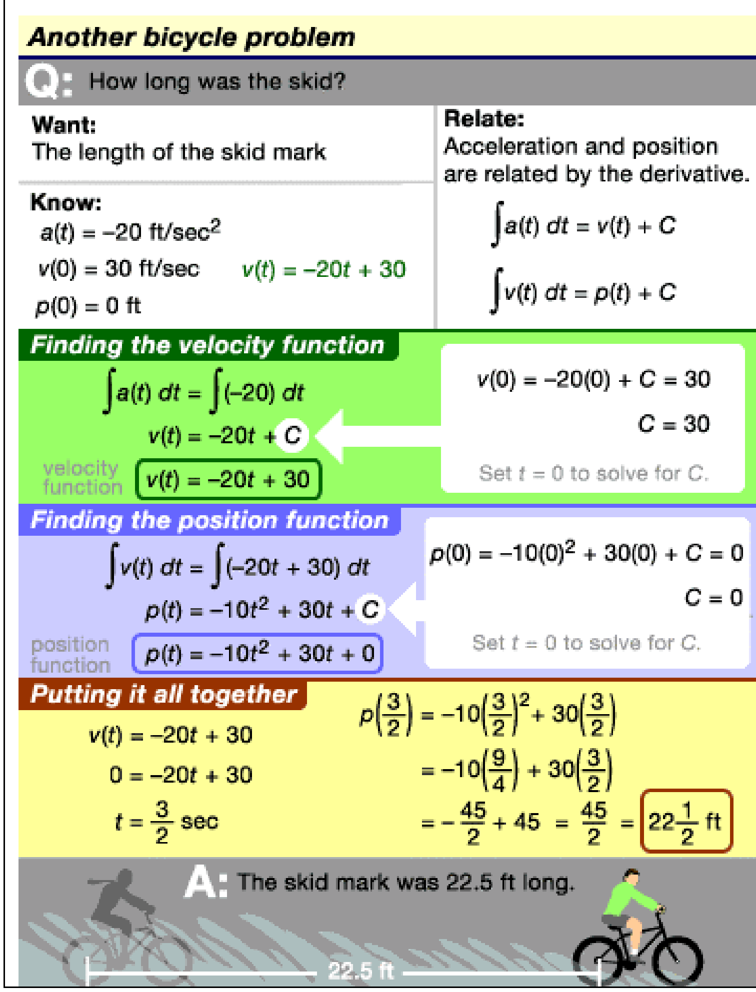
Unit: Applications of Integration **Module:** Motion

Antiderivatives and Motion

key concepts:

- **Position** and motion can be analyzed using calculus.
- **Velocity** is the rate of change of position with respect to time.
  Acceleration is the rate of change of velocity with respect to time.
- Given the velocity function of an object and its position at a specific time, find its position function by taking the antiderivative of velocity and solving for the specific constant of integration.
- Given the acceleration function of an object and its velocity at a specific time, find its velocity function by taking the antiderivative of acceleration and solving for the specific constant of integration.
- An object stops moving when its velocity becomes zero.



Integral calculus empowers you to take an **acceleration** function and deduce the **velocity** and **position** functions.

In this example, Professor Burger states that he was riding his bicycle at 30 ft/sec when he put on the brakes for a constant deceleration of 20 ft/sec^2. Express deceleration as negative acceleration.

Prof. Burger's initial velocity $v(0)$ was 30 ft/sec and his initial position $p(0)$ was 0 ft.

Since acceleration is the derivative of velocity, you can integrate the acceleration function to find the velocity function. You can determine the value of C by using the initial velocity value of 30 ft/sec.

To determine the position function, integrate the velocity function. Once again, use the initial condition you know for position.

Before you can determine the distance Prof. Burger traveled before stopping, you need to know how much time elapsed. The bicycle has come to a stop when the velocity equals 0, so set the velocity function equal to 0 and solve for time t.

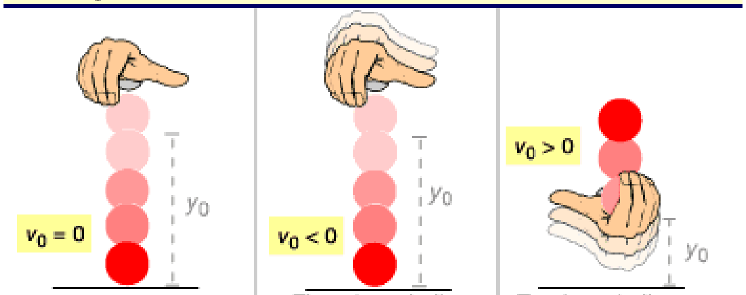
Evaluate the position function for the time of 3/2 sec to arrive at the final position. Since the initial position was defined to be zero, the final position is also the distance traveled, 22.5 ft.

Gravity and Vertical Motion

key concepts:

- The **acceleration** due to gravity of an object is a constant.
- Given the **initial velocity** and **initial position** of an object moving vertically, you can use the fact that the acceleration due to gravity is a constant to find the velocity and position functions.
- The velocity and position functions can be used to answer many questions about the motion of an object.

The equations for vertical motion



Dropping a ball gives it an initial velocity of zero (in the *y*-direction).

Throwing a ball down gives it a negative initial velocity (in the *y*-direction).

Tossing a ball up gives it a positive initial velocity (in the *y*-direction).

$$y(t) = \int v(t)\,dt = \int(-32t + v_0)\,dt = -16t^2 + v_0 t + C \qquad C = y_0$$

$$\boxed{y(t) = -16t^2 + v_0 t + y_0}$$

formula for vertical position

In vertical motion, the **acceleration** due to gravity is a constant 32 ft/sec^2 downward. Therefore acceleration as a function of time can be stated as $a(t) = -32$.

When studying the vertical motion of an object such as a ball, the initial vertical position is denoted y_0. Depending on how the object is released, there are three possibilities for its initial velocity v_0.

Integrating the acceleration function produces the velocity function $v(t) = -32t + v_0$, with v_0 replacing the constant of integration.

Integrating the velocity function produces the position function $y(t)$. Notice that the constant of integration is replaced by the initial position y_0.

The ball problem

Suppose a ball is shot into the air at a rate of 96 ft/sec from the ground.

Q1: How long was the ball in the air? A1: 6 sec

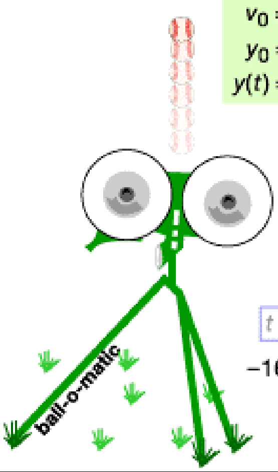
Q2: How high did the ball go?

Q3: What was the velocity of the ball at the instant it hit the ground?

$v_0 = 96$ ft/sec

$y_0 = 0$ ft

$y(t) = -16t^2 + 96t$



WANT: When the ball landed (when $y(t) = 0$)

$$y(t) = -16t^2 + 96t$$

$$-16t^2 + 96t = 0$$

$$t(-16t + 96) = 0$$

$t = 0$ sec The ball was shot up at $t = 0$ sec.

$$-16t + 96 = 0$$

$$t = \frac{96}{16}$$

A1: $t = 6$ sec The ball lands at $t = 6$ sec.

To answer these questions you will first need to write down what you know.

The problem statement tells you that the initial velocity was 96 ft/sec and the initial position was 0 ft.

Use these values in the formula for vertical position.

To find out how long the ball was in the air you need to know when it landed, which is when its position is 0.

There are two solutions. You can discard the solution $t = 0$, since that was the ball left the machine.

The remaining solution gives you the time the ball landed. The ball was in the air for 6 seconds.

Gravity and Vertical Motion

The ball problem

Suppose a ball is shot into the air at a rate of 96 ft/sec from the ground.

Q1: How long was the ball in the air?

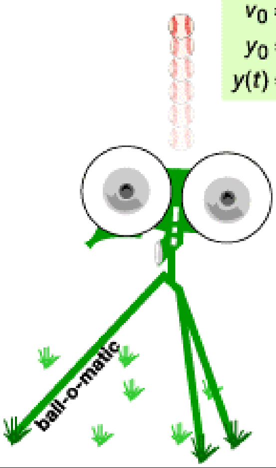
Q2: How high did the ball go?

Q3: What was the velocity of the ball at the instant it hit the ground?

$v_0 = 96$ ft/sec

$y_0 = 0$ ft

$y(t) = 16t^2 + 96t$



WANT: the greatest height the ball attains

$v(t) = -32t + 96$

$0 = -32t + 96$

$t = \dfrac{96}{32}$

$t = 3$ sec

$y(t) = 16t^2 + 96t$

$y(3) = -16(3)^2 + 96(3)$

$= -16(9) + 96(3)$

$= \boxed{144 \text{ ft}}$

The highest the ball gets is 144 ft.

To determine the maximum height, you will need to set the derivative of the position function equal to 0.

In other words, set the velocity function equal to 0.

The result is the time of the maximum height. Evaluate the position function for that value of *t* to determine the maximum height.

Suppose a ball is shot into the air at a rate of 96 ft/sec from the ground.

Q1: How long was the ball in the air? A1: 6 sec

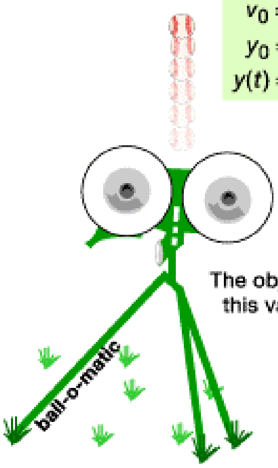
Q2: How high did the ball go? A2: 144 ft

Q3: What was the velocity of the ball at the instant it hit the ground? A3: –96 ft/sec

$v_0 = 96$ ft/sec

$y_0 = 0$ ft

$y(t) = -16t^2 + 96t$



WANT: The velocity of the ball when it lands

Note:

After an object hits the ground it stops. But the object is moving the **instant it reaches the ground.**

The object reaches the ground at $t = 6$ sec, so plug this value into the velocity function.

$v(t) = -32t + 96$

A3: $v(t) = -32(6) + 96$

$\boxed{v(t) = -96 \text{ ft/sec}}$

The final question asks for the velocity of the ball at the instant it struck the ground.

Once the ball has hit the ground it will no longer be moving. Its velocity will be 0. However, this question is asking for the velocity at the instant the ball makes contact with the ground.

You already determined that the ball falls to ground after 6 seconds.

Insert $t = 6$ into the formula for velocity. The velocity is –96 ft/sec. Recall that the negative sign means that the ball was traveling downward.

Unit: Applications of Integration **Module:** Motion

Solving Vertical Motion Problems

key concepts:

- Given the **initial velocity** and **initial position** of an object moving vertically, you can use the fact that the acceleration due to gravity is a constant to find the velocity and position functions.
- The velocity and position functions can be used to answer many questions about the motion of an object.

The falling book problem

Suppose a book is dropped into a well and the book landed 5 seconds later.

Q: How deep is the pit?

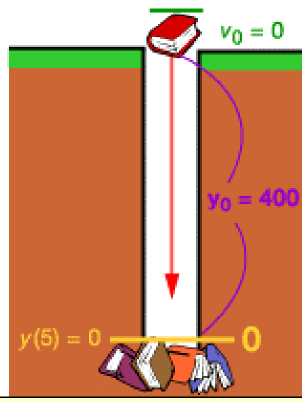
WANT: the distance to the bottom of the pit (y_0)

KNOW: $v_0 = 0$

$$y(t) = -16t^2 + y_0$$
$$y(5) = 0$$

$$y(5) = -16(5)^2 + y_0$$
$$-16(25) + y_0 = 0$$
$$-400 + y_0 = 0$$
$$y_0 = 400 \text{ ft}$$

A: The pit is 400 ft deep.



This problem asks you to use the length of time an object was falling in order to determine the distance it fell.

A falling object has no **initial velocity** since it is simply released, not thrown.

This fact simplifies the position function.

You also know that the height when $t = 5$ is 0.

Evaluate the function at $t = 5$ to set up an equation you can solve.

Solve for the initial velocity y_0 to determine how far the object fell, which is also the depth of the pit.

The throwing the ball upwards problem

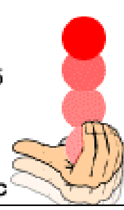
Q: How fast must an object be thrown upwards from the ground for it to reach a maximum height of 25 ft?

WANT: the initial velocity (v_0)

KNOW: $y_0 = 0$

$$y(t) = -16t^2 + v_0 t$$
$$v(t) = -32t + v_0$$
$$0 = -32t + v_0$$
$$t = \frac{v_0}{32} \text{ sec}$$

$$y(t) = -16t^2 + v_0 t$$
$$-16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 25$$
$$-16\,\frac{v_0^2}{32^2} + \frac{v_0^2}{32} = 25$$
$$-16\,\frac{v_0^2}{32^2} + 32\,\frac{v_0^2}{32^2} = 25$$
$$16\,\frac{v_0^2}{32^2} = 25$$
$$v_0^2 = \frac{32^2}{16} \cdot 25$$
$$v_0 = \frac{32}{4} \cdot 5$$
$$v_0 = 40 \text{ ft/sec}$$



A: The object must be thrown upwards at a velocity of 40 ft/sec.

In this situation you are asked to find the initial velocity needed to attain a given maximum height.

First you must find an expression for the time at which the object attains its maximum height. This will happen when the velocity is zero, so set the velocity function equal to 0 and solve for t in terms of v_0.

Then substitute the expression for t into the position function and. Set the result equal to the maximum height of 25 and solve for v_0. You will need to use your techniques for simplifying algebraic expressions to arrive at the answer.

An initial velocity of 40 ft/sec will give the object a maximum height of 25ft.