Unit: Curve Sketching **Module:** Graphing Using the Derivative

Graphs of Polynomial Functions

key concepts:

- To graph a function:
  1. Find critical points using the first derivative.
  2. Determine where the function is increasing or decreasing.
  3. Find inflection points using the second derivative.
  4. Determine where the function is concave up or concave down.

- On an interval, the sign of the first derivative indicates whether the function is increasing or decreasing. The sign of the second derivative indicates whether the function is concave up or concave down.

Using the first derivative

Sketch the graph of $f(x) = x^4 - 10x^3 + 5$.

Step 1 Find the derivative of $f(x)$ and the **critical points**.

$f'(x) = 4x^3 - 30x^2$

$4x^3 - 30x^2 = 0$ Set $f'(x) = 0$.

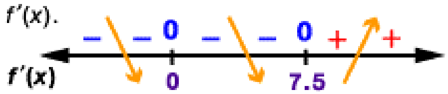
$2x^2(2x - 15) = 0$

$2x^2 = 0$ or $2x - 15 = 0$

$x = 0$ or $x = 7.5$ only **critical points**

remember: Polynomials are defined for all real numbers.

Step 2 Make a sign chart for $f'(x)$.



Sketching an accurate drawing of a function requires several steps, even for a polynomial function.

First you need to take the derivative of the function so you can determine its critical points. Set the derivative equal to zero and solve for x. This function has two critical points.

Next, make a sign chart for the derivative. Choose a point from each interval to determine the sign of the derivative at that point. Then use arrows to indicate whether the function is increasing or decreasing. This function has a minimum at $x = 7.5$.

Using the second derivative

Step 3 Find the second derivative of $f(x)$ and **inflection points**.

$f''(x) = 12x^2 - 60x$

$12x^2 - 60x = 0$

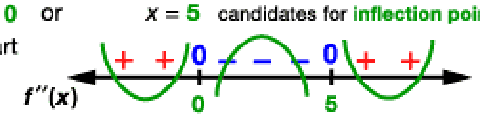
$12x(x - 5) = 0$

$12x = 0$ or $x - 5 = 0$

$x = 0$ or $x = 5$ candidates for **inflection points**

Inflection points at $x = 0$ and $x = 5$.

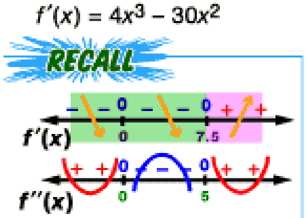
Step 4 Make a sign chart for $f''(x)$.

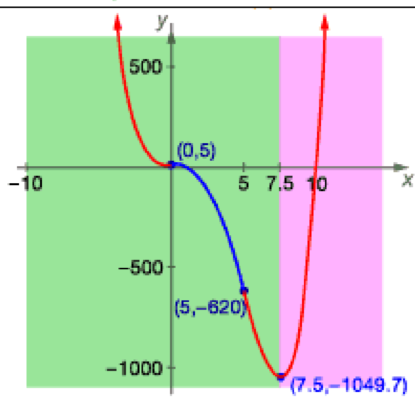


Third, take the second derivative of the function in order to check for possible inflection points. Set the second derivative equal to zero and solve for x.

Fourth, make a sign chart to determine if the inflection point candidates are inflection points or not. Choose an x-value from each interval to determine the sign of the second derivative and the concavity of the function on that interval. Points where the function changes concavity are inflection points. This function has two inflection points.

$f'(x) = 4x^3 - 30x^2$





Finally, use the information from the sign charts to sketch the graph of the function. The graph is decreasing for all points to the left of $x = 7.5$ and increasing to the right.

The graph is concave up the left of $x = 0$ and to the right of $x = 5$. It is concave down in between. Therefore (0, 5) and (7.5, –1049.7) are points of inflection.

Unit: Curve Sketching **Module:** Graphing Using the Derivative

Cusp Points and the Derivative

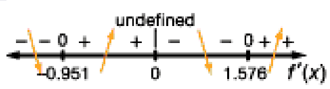
key concepts:

- Functions with fractional exponents could potentially have **cusp points**. A cusp point is a point where the curve abruptly changes direction.
- To graph a function:
  1. Find critical points using the first derivative.
  2. Determine where the function is increasing or decreasing.
  3. Find inflection points using the second derivative.
  4. Determine where the function is concave up or concave down.

Challenge: Sketch the graph of $f(x) = x^{2/3}(x^2 - x - 6)$.

Step 1 $f'(x) = \dfrac{8x^2 - 5x - 12}{3x^{1/3}}$ critical points: $x \approx -0.951$, $x = 0$, $x \approx 1.576$

Step 2 Make a sign chart for $f'(x)$.



Step 3 Find the second derivative of $f(x)$ and inflection point candidates.

$$f''(x) = \frac{3x^{1/3}(16x - 5) - (8x^2 - 5x - 12)(x^{-2/3})}{9x^{2/3}}$$ Use the quotient rule.

$$3x^{1/3}(16x - 5) - (8x^2 - 5x - 12)(x^{-2/3}) = 0$$

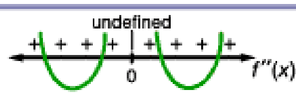
$$48x^2 - 15x - 8x^2 + 5x + 12 = 0$$

$$40x^2 - 10x + 12 = 0$$

There are no real solutions.

inflection point candidate: f'' is undefined at $x = 0$

Step 4 Make a sign chart for $f''(x)$.



Expect to see strange behavior in the graphs of functions with fractional exponents.

First, find critical points by setting the derivative equal to zero and solving for x. The derivative is undefined at $x = 0$ even though the function exists, so it is a critical point, too.

Next make a sign chart for the derivative, including arrows for the behavior of the function and labels for the critical points.

Third, find possible inflection points by setting the second derivative equal to zero and solving for x.

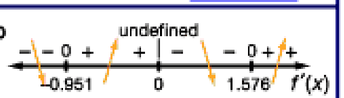
In this case, the second derivative is never equal to zero. However, it is undefined for $x = 0$, which is an inflection point candidate.

Fourth, make a sign chart for the second derivative. Since the concavity does not change at $x = 0$, it is not an inflection point.

Summit: $f(x) = x^{2/3}(x^2 - x - 6)$

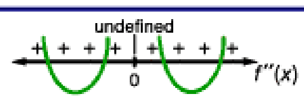
Step 1 critical points: $x \approx -0.951$, $x = 0$, $x \approx 1.576$

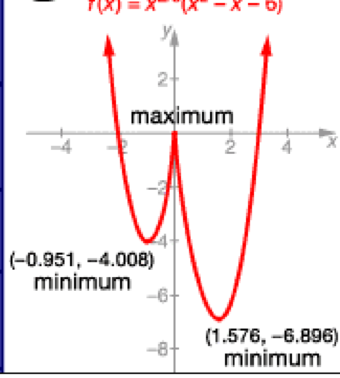
Step 2



Step 3 Inflection point candidate $x = 0$

Step 4





Finally, assimilate the information from the sign charts to sketch the graph of the function. From left to right the graph first decreases, then increases, then decreases, and finally increases.

Since the derivative is undefined at $x = 0$, the line tangent to the curve is vertical. This forms a sharp turn, called a **cusp point**, at (0, 0).

On either side of the cusp point the function is concave up. There is no inflection point.

Unit: Curve Sketching **Module:** Graphing Using the Derivative

Domain-Restricted Functions and the Derivative

key concepts:

- If the variable of a function is raised to a fractional exponent that has an even denominator, the function may not be defined for all real numbers.
- To graph a function:
  1. Find critical points using the first derivative.
  2. Determine where the function is increasing or decreasing.
  3. Find inflection points using the second derivative.
  4. Determine where the function is concave up or concave down.
- On an interval, the sign of the first derivative indicates whether the function is increasing or decreasing. The sign of the second derivative indicates whether the function is concave up or concave down.

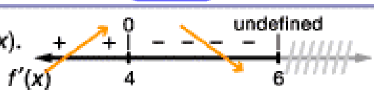
The domain of a function with a fractional exponent

Challenge: Consider $f(x) = x(6-x)^{1/2}$.

$f(x)$ is only defined for $x \le 6$.

Step 1 $f'(x) = \dfrac{12-3x}{2(6-x)^{1/2}}$ critical points $x = 4$, $x = 6$

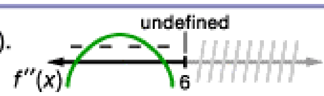
Step 2 Make a sign chart for $f'(x)$.



Step 3 Find the second derivative of $f(x)$ and inflection point candidates.

$$f''(x) = \frac{3x-24}{4(6-x)^{3/2}}$$

Step 4 Make a sign chart for $f''(x)$.



If a function involves a fractional exponent that has an even denominator even when reduced, then the domain of the function may be restricted. This particular function is not defined for x-values greater than 6. Keep this in mind as you go through the steps for sketching its graph.

First take the derivative using the product rule. Set it equal to zero to find the critical points. Then make a sign chart for first derivative. It is not defined for x-values greater than or equal to 6.

Third, take the second derivative, which requires the quotient rule. Set it equal to zero and solve for x. The only solution is $x = 8$, but that is not in the domain, so there are no inflection point candidates.

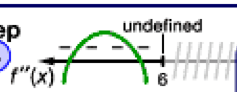
Fourth, make a sign chart for the second derivative.

Step 1 critical points $x = 4$, $x = 6$

Step 2

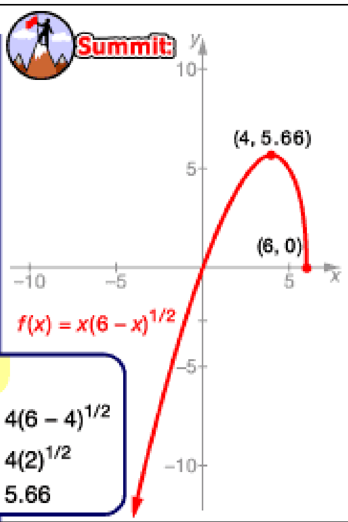
Step 3 There are no inflection points.

Step 4



Some values of the function (6, 0) (4, 5.66)

Summit:



Let $x = 4$.

$$\begin{aligned} f(4) &= 4(6-4)^{1/2} \\ &= 4(2)^{1/2} \\ &\approx 5.66 \end{aligned}$$

Once you have analyzed the function, sketch its graph.

The function is increasing to the left of $x = 4$ and decreasing from $x = 4$ to $x = 6$. It has a maximum at $x = 4$. There is an endpoint minimum at $x = 6$ because the function is decreasing at the end of its domain.

The graph is concave down everywhere, so there is no inflection point.

Calculus Lecture Notes

thinkwell
©2001

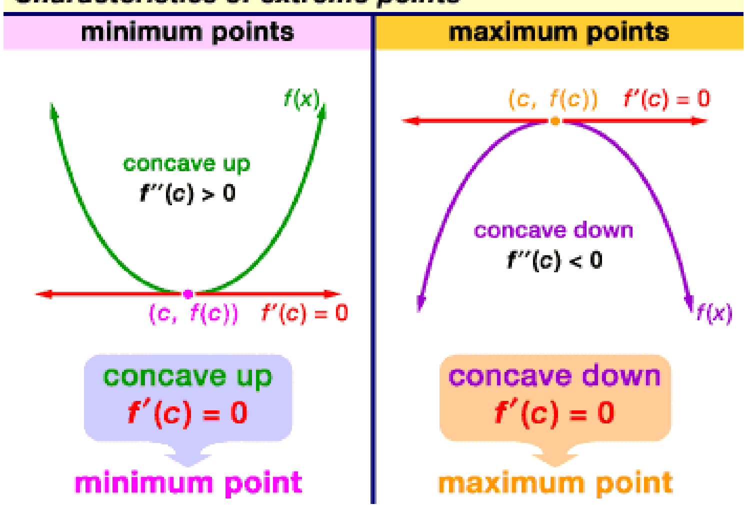
Unit: Curve Sketching **Module:** Graphing Using the Derivative

The Second Derivative Test

key concepts:

- If the graph of a function has a tangent line with a slope of 0 and the graph is concave up at the same point, then the point is a minimum point of the function. If the graph of the function is concave down at that point, then the point is a maximum point of the function.
- If the graph of a function has a tangent line with a slope of 0 and the second derivative is at that point is also 0, then the second derivative test is inconclusive.
- The **second derivative test** states that if $f'(c) = 0$ and the second derivative of f exists on an open interval containing c, then $f(c)$ can be classified as follows.
  1) If $f''(c) > 0$, then $f(c)$ is a relative minimum of f.
  2) If $f''(c) < 0$, then $f(c)$ is a relative maximum of f.
  3) If $f''(c) = 0$, then the test is inconclusive.

Characteristics of extreme points



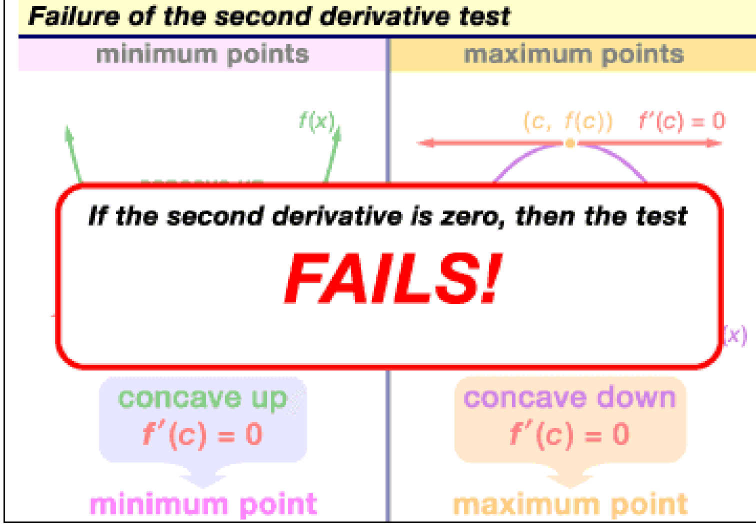
The **second derivative test** indicates whether a critical point is a maximum point or minimum point without making a sign chart for the first derivative.

To use this test, you must set the first derivative equal to zero and find the x-values that make it equal to zero or undefined. These are the critical points

In order for a critical point to be a minimum, the graph must be concave up at that point, as shown on the far left. Therefore, if the second derivative is positive, then the critical point is a minimum point.

On the other hand, for a critical point to be a maximum, the graph must be concave down at that point. Therefore, if the second derivative is negative then the critical point is a minimum point.

Failure of the second derivative test



The third case occurs when the second derivative equals zero at the critical point. In this case the second derivative test fails. The critical point might correspond to a maximum point, a minimum point or neither. To find out, you will need to make a sign chart for the first derivative.

The second derivative test may not save you much work if you are trying to sketch the graph of a function. However, it may be useful if you are solving a problem involving maximization or minimization.