**Unit:** Practical Application of the Derivative **Module:** Linear Approximation

# Higher-Order Derivatives and Linear Approximations

**key concepts:**

- You can find successive derivatives of a function by differentiating each result.
- Derivatives can allow you to find a **linear approximation** for values of complicated functions near values you know.

## *Finding multiple derivatives*

Consider $f(x) = x^4 - 3x^2 + 1$.

$f'(x) = 4x^3 - 6x$ The derivative of $f(x)$ is the first derivative.

$f''(x) = 12x^2 - 6$ The derivative of the first derivative is the second derivative.

$f'''(x) = 24x$ The derivative of the second derivative is the third derivative.

$f^{(4)}(x) = 24$ The derivative of the third derivative is the fourth derivative.

A **higher-order derivative** is the derivative of a derivative. You can take as many higher-order derivatives as you like. In fact, some applications of calculus will require you to take an infinite number of higher-order derivatives.

The original derivative is called the **first derivative**. Each successive derivative is numbered one higher, so the next is the **second derivative** and the following is the **third derivative**.

## *Multiple derivatives and Leibniz notation*

Consider $y = x^3 - 2x^4 + 3$.

$\frac{dy}{dx} = 3x^2 - 8x^3$ "the derivative of $y$"

$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[3x^2 - 8x^3\right]$

$\frac{d^2y}{dx^2} = 6x - 24x^2$ "the second derivative of $y$ with respect to $x$"

$\frac{d^3y}{dx^3} = 6 - 48x$ "the third derivative of $y$ with respect to $x$"

Higher-order derivatives can also be described using Leibniz notation.

To indicate the second derivative of $y$ with respect to $x$, put a superscripted 2 over the $d$ in the numerator and over the $y$ in the denominator.

Successive derivatives follow the same pattern.

## *Approximating function values with derivatives*

**Challenge:** What is the value of $\sqrt{4.1}$?

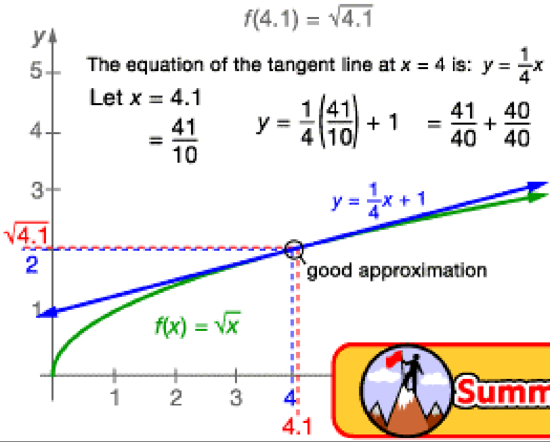
Let $f(x) = \sqrt{x}$. **Want:** $f(4.1)$ **Know:** $f(4) = \sqrt{4}$

$f(4.1) = \sqrt{4.1}$ $= 2$

The equation of the tangent line at $x = 4$ is: $y = \frac{1}{4}x + 1$

Let $x = 4.1$ $= \frac{41}{10}$

$y = \frac{1}{4}\left(\frac{41}{10}\right) + 1 = \frac{41}{40} + \frac{40}{40} = \frac{81}{40}$




**Summit:** $\sqrt{4.1} \approx \frac{81}{40}$

The line tangent to a curve can be used to approximate values of the function. The tangent line is a good approximation close to the point of tangency because the tangent line behaves like the curve near that point and because lines are very easy to evaluate.

Using a line to estimate the value of a more complicated function is called a **linear approximation**.

To make a linear approximation, find the equation of the line tangent to the complicated curve at a value for the curve that you can evaluate and close to the unknown value.

Once you have the equation of the line, plug the $x$-value of the unknown point into the equation of the line. The result is a good approximation of the original function.

# Calculus Lecture Notes

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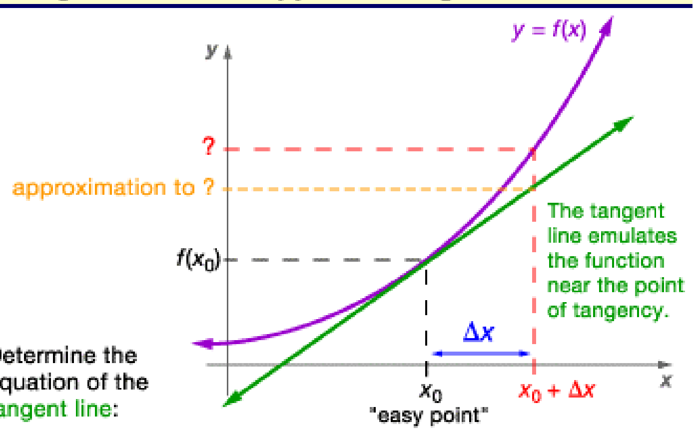
**Unit:** Practical Application of the Derivative **Module:** Linear Approximation

## Using the Tangent Line Approximation Formula

**key concepts:**

- The **tangent line approximation formula**: $f(x+\Delta x) \approx f(x) + f'(x)(\Delta x)$

### *Deriving a formula for approximating functions*




Determine the equation of the tangent line:

$y - y_1 = m(x - x_1)$ Use the point-slope form.

$y - f(x_0) = f'(x_0)(x - x_0)$ $(x_0, f(x_0))$ is a point on the line.

$y = f'(x_0)(x - x_0) + f(x_0)$ $f'(x_0)$ is the slope of the line.

Evaluate the formula for the line at the desired point:

$y = f'(x_0)(x_0 + \Delta x - x_0) + f(x_0)$

$\boxed{y = f'(x_0)(\Delta x) + f(x_0)}$ Simplify.

The process of finding a **linear approximation** can be described by a general formula.

When making a linear approximation, you start by finding the equation of the line tangent to the curve at an "easy point." A point is considered easy if you can evaluate the function at that point. For example, the square root function is easy to evaluate at the number 9.

The distance between the easy point and the point you are interested in is the change in $x$, or $\Delta x$.

Use the point-slope form of a line. Notice that the slope of the tangent line is equal to the first derivative of the curve evaluated at the "easy point."

The $y$-value of this equation tells you the height of the line at that $x$-point. Remember, this $y$-value is a good approximation for the function at that point. This equation is called the **tangent line approximation formula**.

### *Applying the linear approximation formula*

Approximate $\sqrt[3]{7.9}$.

Let $f(x) = \sqrt[3]{x}$.

**Want:** $f(7.9)$

**Know:** $f(8) = \sqrt[3]{8}$
$= 2$

**Cast of Characters**

$x = 8$

$x + \Delta x = 7.9$

$\Delta x = -0.1$

**Find the derivative.**

$f(x) = \sqrt[3]{x}$

$= x^{1/3}$ Write the radical as an exponent.

$f'(x) = \frac{1}{3}x^{-2/3}$ Use the power rule.

$= \frac{1}{3x^{2/3}}$

$= \frac{1}{3(\sqrt[3]{x})^2}$

$f(7.9) \approx f(8) + \frac{1}{3(8)^{2/3}}(-0.1)$

$= 2 + \frac{1}{3(4)}(-0.1)$

$= 2 + \frac{1}{12}\left(\frac{-1}{10}\right)$

$= 2 - \frac{1}{120} = \frac{240}{120} - \frac{1}{120}$

$\boxed{\sqrt[3]{7.9} \approx \frac{239}{120}}$

To use the tangent line approximation formula, start by finding a good easy point and the distance between the easy point and the point you want to approximate.

In this example you are asked to approximate the value of the cube root of 7.9. Since you know that the cube root of 8 is 2, you can use that point as your easy point. The signed distance between 8 and 7.9 is –0.1.

Find the derivative of the cube root function using the power rule. Evaluate the derivative at the "easy point." Finally, plug all of that information into the linear approximation formula.

The resulting approximation is accurate to 4 decimal places.

# Calculus Lecture Notes

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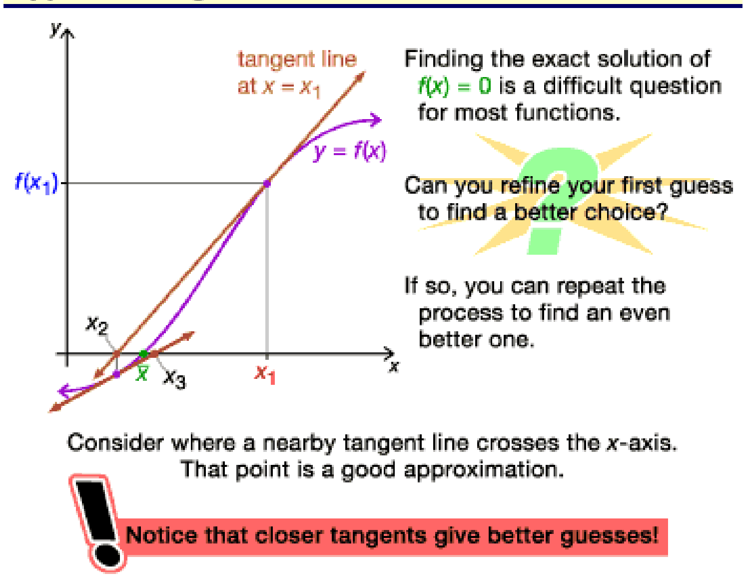
## Newton's Method

key concepts:

- **Newton's method** iterates the approximation process and thereby finds successively better approximations for the solution to a function.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### Approximating solutions of functions



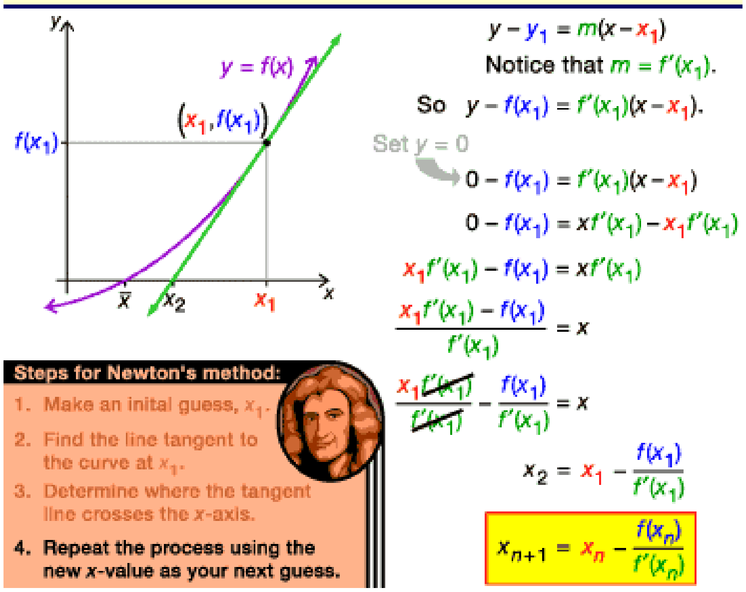

The tangent line can be used to approximate solutions of functions.

Graphically, a solution to a function is the $x$-value where the graph of the function crosses the $x$-axis. Finding these solutions is actually very complicated. Formulas exist for second- and third-order polynomials, but there are many more functions that you might want to solve.

However, it is easy to find the point where a line crosses the $x$-axis. Notice also that the line tangent to a curve can be a decent approximation.

More importantly, if you repeat the process of finding the tangent line you can refine your guess and get closer and closer to the actual solution.

### Newton's method




Sir Isaac Newton noticed that tangent lines could be used to approximate the solutions to functions. More importantly, he noticed that repeating the process could refine that approximation. This process is known as **Newton's method**.

Newton's method starts with a guess. Once you have selected the guess, find the equation of the line tangent to the curve at that point.

Setting the $y$-value of that equation equal to 0 solves for where the line crosses the $x$-axis.

The value where the curve crosses the $x$-axis is often a better guess than the original.

Once you have found that second guess, you can repeat the process with that new point and find an even better guess.

However, Newton's method does require that the curve behave a specific way. In some cases, Newton's method will produce successively worse guesses.