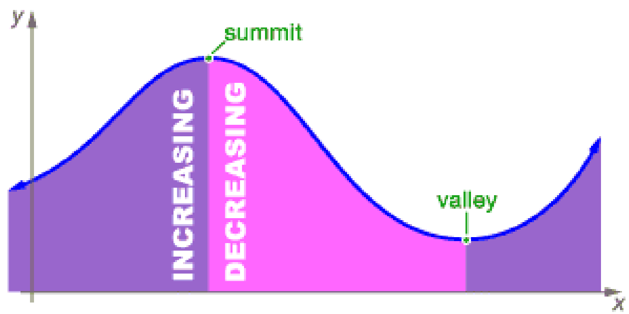
Connection Between Slope and Optimization

key concepts:

- On an interval, the sign of the derivative of a function indicates whether that function is increasing or decreasing.
- Values that make the derivative of a function equal to 0 are candidates for the location of **maxima** and **minima** of the function.

Tangent lines and the behavior of a curve



Q: How can you determine where the curve is increasing or decreasing?

A: Study the tangent lines.

Q: What if the derivative equals 0?

A: The function might not be increasing or decreasing.

tangent line	function	
positive slope	increasing	$f'(c) > 0$
negative slope	decreasing	$f'(c) < 0$

If $f'(c) = 0$ then $f(c)$ is a candidate for a maximum or minimum of f.

The behavior of the tangent line gives insight into the behavior of a curve.

Notice that for an interval where the curve is increasing, the tangent lines must be positively sloped. Likewise, when the curve is decreasing, the tangent lines must be negatively sloped.

In addition, when a curve changes from increasing to decreasing or from decreasing to increasing, the curve can attain a **maximum** or **minimum** value. These points occur where the derivative is equal to 0 (or undefined).

A cubic function

Consider: $f(x) = x^3 + x^2 - 10x + 1$

Take the derivative of f.

$$f'(x) = 3x^2 + 2x - 10 \quad \text{derivative}$$

Let $x = 0$.

$$f'(0) = 3(0)^2 + 2(0) - 10$$
$$= -10 \quad f \text{ is decreasing around } x = 0.$$

Let $x = 10$.

$$f'(10) = 3(10)^2 + 2(10) - 10$$
$$= 3(100) + 20 - 10$$
$$= 310 \quad f \text{ is increasing around } x = 10.$$

tangent line	function	
positive slope	increasing	$f'(c) > 0$
negative slope	decreasing	$f'(c) < 0$

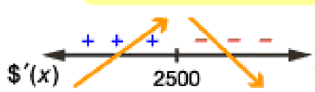
If $f'(c) = 0$ then $f(c)$ is a candidate for a maximum or minimum of f.

Here is an example of this concept in action.

Notice that the derivative is negative for $x = 0$. However the derivative is positive for $x = 10$.

Not only does this give you insight to the shape of the curve, it also indicates that there is probably a minimum point between these two x-values.

Connection Between Slope and Optimization

Maximizing profit **Profit:** $\$(x) = \dfrac{5000x - x^2}{50{,}000}$ where x is measured in square feet Q What size factory will maximize profit? Take the derivative of *f*. $\$'(x) = \dfrac{1}{10} - \dfrac{1}{25{,}000}x$ Set the derivative equal to 0 and solve for x. $\$'(x) = 0$ $\dfrac{1}{10} - \dfrac{1}{25{,}000}x = 0$ $-\dfrac{1}{25{,}000}x = -\dfrac{1}{10}$ $x = \dfrac{25{,}000}{10}$ $x = 2500\ \text{ft}^2$ candidate for a maximum or minimum	One application of this process is called **optimization**. If you can find the maximum or minimum value of a function, then you can determine where a function will produce the greatest profit or the least amount of waste. Start by finding the derivative of the function. Next set the derivative equal to 0 and solve for x.
Let $x = 1000$. $\$'(1000) = \dfrac{1}{10} - \dfrac{1000}{25{,}000}$ $= \dfrac{1}{10} - \dfrac{1}{25}$ $\dfrac{1}{25}$ is less than $\dfrac{1}{10}$ positive Let $x = 5000$. $\$'(5000) = \dfrac{1}{10} - \dfrac{5000}{25{,}000}$ $= \dfrac{1}{10} - \dfrac{1}{5}$ $\dfrac{1}{5}$ is larger than $\dfrac{1}{10}$ negative  maximum at $x = 2500$	Once you know the candidates for the maximum or minimum points, you can test the derivative on either side of those candidates to see if the function is increasing or decreasing on that side. If a function is increasing to the left and decreasing to the right of a candidate, then there is a maximum value at that point. If a function is decreasing to the left and increasing to the right of a candidate, then there is a minimum value at that point. It is a good idea to make some sort of chart for this information so you will remember where the function increases and decreases.
Q What is the maximum profit? A $125 $\$(2500) = \dfrac{\cancel{5000}^{10}(25\cancel{00})}{\cancel{50{,}000}} - \dfrac{(\cancel{2500}^{5})(25\cancel{00})}{\cancel{50{,}000}}$ $= 250 - 125$ $= \boxed{125}$	Once you know where the maximum value is located, you can plug that value into the function to find the actual maximum value.

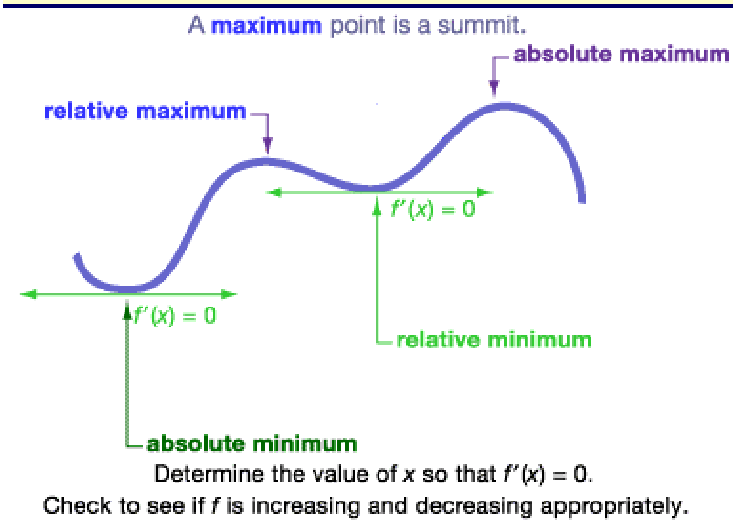
Unit: Practical Application of the Derivative **Module:** Optimization

The Fence Problem

key concepts:

- Values that make the derivative of a function equal to 0 or undefined are candidates for **maxima** and **minima** of the function.
- The fence problem involves maximizing the fenced-in area without changing the amount of fence used. Set the derivative of the area function equal to 0 and solve.

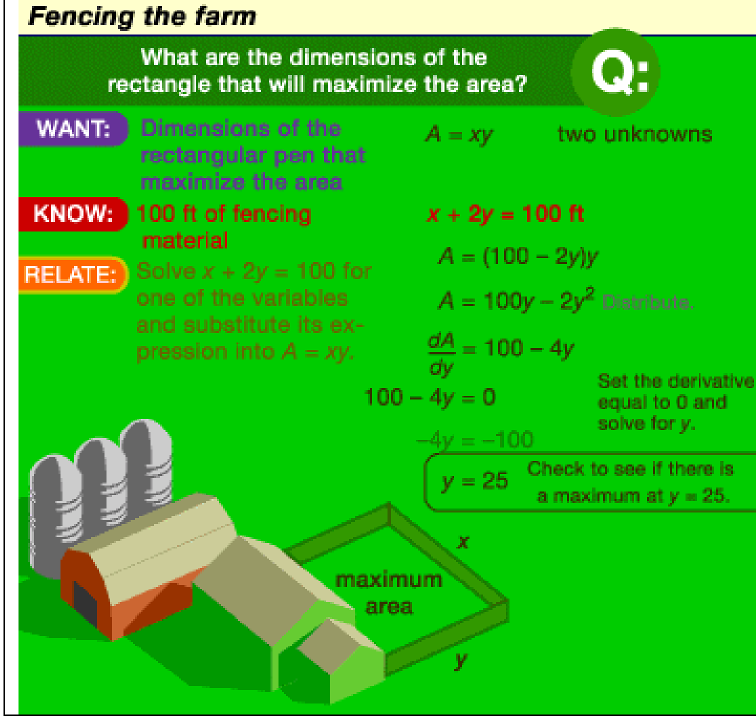
Relative minima vs absolute minima



There are actually two types of **maxima** and **minima**. A **relative maximum** occurs anytime you have a summit, but it does not have to be that absolute highest point the graph attains. The **absolute maximum** is the highest the graph can get.

Likewise, a **relative minimum** occurs anytime you have a valley. The **absolute minimum** is the lowest the graph can get.

Fencing the farm



When solving a problem, think about what you want, figure out what you know, and relate the two.

The question asks you to find the dimensions of the fence that would enclose the greatest area, using the farmhouse as one side of the pen. So the question asks for you to maximize the area.

You are told that you have 100 feet of fencing material. You also know that the material makes up three sides of the perimeter of the rectangle.

By solving the perimeter equation for x you can substitute that expression into the area equation. This relates what you want and what you know. Once you have the area equation in terms of a single variable you can differentiate.

Set the derivative of the area equation equal to 0 to find potential maximum points.

Be careful which variable you use when you solve this problem. Do not mix them up!

Unit: Practical Application of the Derivative **Module:** Optimization

The Box Problem

key concepts:

- The box problem involves maximizing the volume of an open box constructed from a given rectangular sheet of material.

Maximizing the volume

WANT: The value of s such that the volume of the box is **maximized**

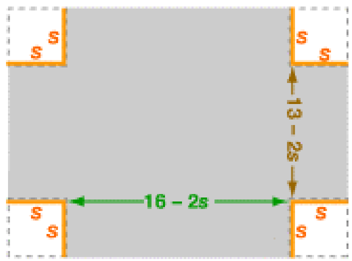
KNOW: Dimensions are 16 in. by 13 in.

$V = bwh$

RELATE: $V = (16 - 2s)(13 - 2s)(s)$

$= (208 - 58s + 4s^2)s$

$= 208s - 58s^2 + 4s^3$



$$\frac{dV}{ds} = 208 - 116s + 12s^2 = 0$$

$104 - 58s + 6s^2 = 0$ Divide both sides by 2.

$6s^2 - 58s + 104 = 0$ Write in standard form.

$$s = \frac{-(-58) \pm \sqrt{(-58)^2 - 4(6)(104)}}{2(6)}$$ Substitute into the quadratic formula.

There are two solutions. $s = \dfrac{58 \pm \sqrt{868}}{12}$

$s = \dfrac{58 + \sqrt{868}}{12}$ or $s = \dfrac{58 - \sqrt{868}}{12}$

$s = \dfrac{58 + 29.46}{12}$ or $s = \dfrac{58 - 29.46}{12}$

$s \approx 7.288$ or $s \approx 2.378$

Q: What size square should be removed from each corner in order to create a box that maximizes volume?

In the box problem, you are asked to maximize the volume of a box constructed from a rectangular sheet.

You know the dimensions of the sheet and the formula for volume of the box.

Relate the dimensions and volume by expressing each of the dimensions in terms of the length of a side of the square that must be removed from each of the corners. The resulting volume equation contains only one variable.

Set the derivative of the volume equal to 0 to find maximum point candidates. Notice that the values of s represent the lengths of the sides of the squares that must be removed to maximize the volume.

The quadratic formula gives two possible maximum candidates.

WANT: The value of s such that the volume of the box is **maximized**

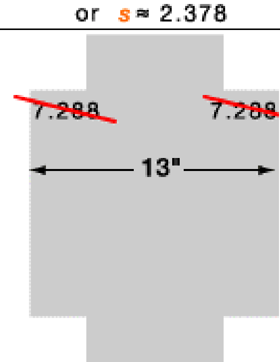
KNOW: Dimensions are 16 in. by 13 in.

$V = bwh$

RELATE: $V = (16 - 2s)(13 - 2s)(s)$

$= (208 - 58s + 4s^2)s$

$= 208s - 58s^2 + 4s^3$

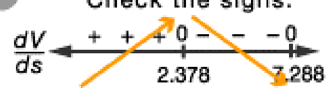


$$\frac{dV}{ds} = 208 - 116s + 12s^2 = 0$$

Which of the critical points gives the maximum volume?

$s \approx \cancel{7.288}$ or $s \approx 2.378$

Check the signs.



Let $s = 0$. $\dfrac{dV}{ds} = 208 - 116(0) + 12(0)^2$

$= 208$ positive

For $s = 5$. $\dfrac{dV}{ds} = 208 - 116(5) + 12(5)^2$

$= 208 - 580 + 300$

$= -72$ negative

Q: What size square should be removed from each corner in order to create a box that maximizes volume?

The greater of the two maximum candidates does not make sense. Removing this length from the dimensions of the box would result in negative length. Therefore you only have one maximum candidate.

Make sure the candidate is a maximum by checking the sign of the derivative on either side of the point.

Notice that to the left of the point the derivative is positive. The function is increasing.

To the right the derivative is negative, so the function is decreasing.

If you increase to the left and decrease to the right, then the point in between must be a maximum.

Make sure that you answer the question! In this problem you were asked to find the dimensions of the square that must be removed from the corners. The square is 2.378 by 2.378. If you were asked to find the dimensions of the box the answer would be different.

Unit: Practical Application of the Derivative **Module:** Optimization

The Can Problem

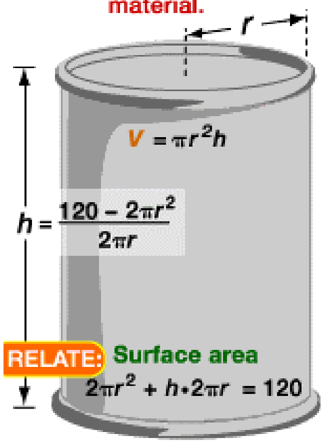
key concepts:

- Values that make the derivative of a function equal to 0 or undefined are candidates for maxima and minima of the function.
- The can problem involves maximizing the volume of a cylindrical can constructed from a given quantity of material.

Setting up the problem

WANT: The dimensions of the can such that its volume is maximized

KNOW: You have 120 in^2 of material.



Solve for h.

$$h \cdot 2\pi r = 120 - 2\pi r^2$$

$$h = \frac{120 - 2\pi r^2}{2\pi r}$$

Find the volume:

$$V = \cancel{\pi}\, r^{\cancel{2}}\left(\frac{\overset{60}{\cancel{120}} - \cancel{2}\pi r^2}{\cancel{2\pi r}}\right)$$

$$= r(60 - \pi r^2) \quad \text{Cancel.}$$

$$= 60r - \pi r^3 \quad \text{Distribute.}$$

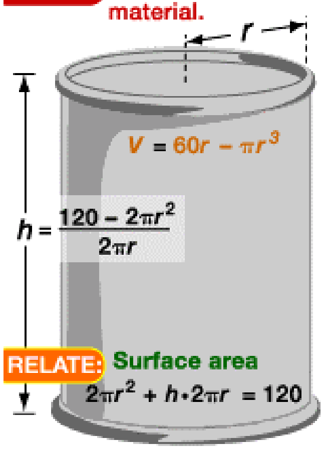
Here you are asked to find the dimensions of a can made from a certain amount of material in a way that maximizes the volume of the can.

You know the surface area of the can as well as the formula for the volume of a cylinder.

By solving the surface area for h, you can relate the surface area and the volume. Substituting gives you an equation for volume in terms of the radius of the can.

WANT: The dimensions of the can such that its volume is maximized

KNOW: You have 120 in^2 of material.



Set the derivative equal to 0 and solve for r.

$$\frac{dV}{dr} = 60 - 3\pi r^2 = 0$$

$$-3\pi r^2 = -60$$

$$r^2 = \frac{60}{3\pi}$$

$$r^2 = \frac{20}{\pi}$$

Remember: Length is positive.

$$r = \sqrt{\frac{20}{\pi}} \quad \text{or} \quad \cancel{r = -\sqrt{\frac{20}{\pi}}}$$

maximum at

$$\boxed{r = \sqrt{\frac{20}{\pi}} = 2.5231325\ldots}$$

maximum at

$$\boxed{h = \frac{20}{\sqrt{5\pi}} = 5.0462650\ldots}$$

Once you have the formula for the volume you can differentiate.

Set the derivative of the volume equal to 0 and solve for r.

Since the radius cannot be negative, one of the answers can be rejected.

Checking the derivative to the left and the right of the maximum candidate will show that the volume increases to the left and decreases to the right. So the candidate does correspond to a maximum.

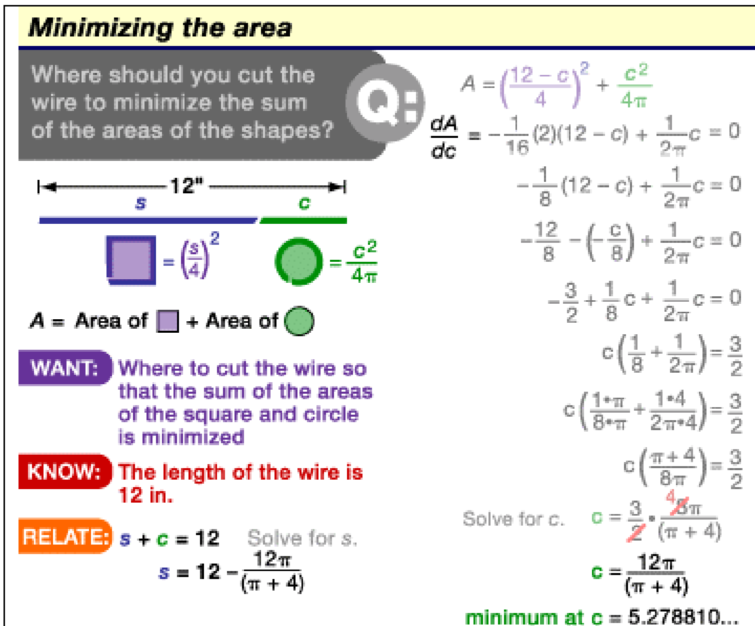
The question asked for the dimensions of the can, not just the radius. Make sure that you give both dimensions.

Unit: Practical Application of the Derivative **Module:** Optimization

The Wire Cutting Problem

key concepts:

- Values that make the derivative of a function equal to 0 or undefined are candidates for maxima and minima of the function.
- The wire cutting problem involves minimizing the sum of the areas of a square and a circle formed from a fixed length of material.
- For a given perimeter, a circle encloses more area than a square does.



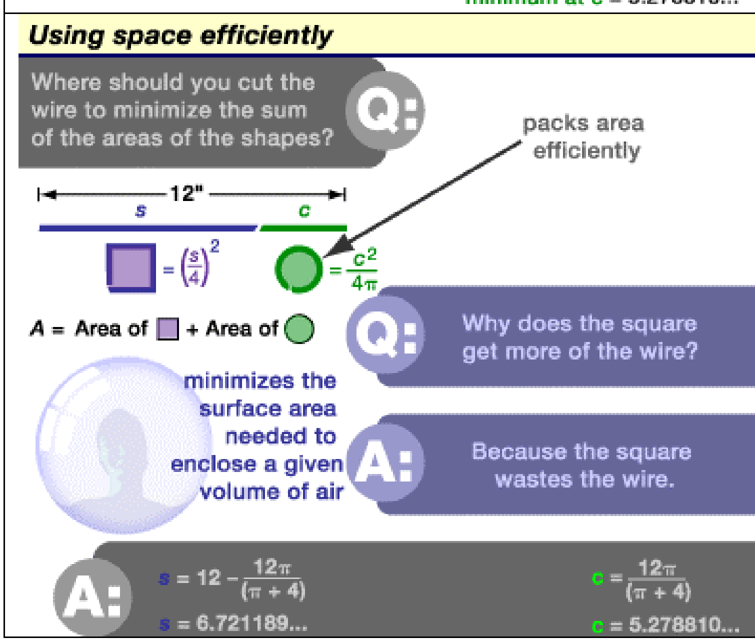
In this problem you are asked to find where a wire should be cut so that the area made by the resulting two pieces is minimized.

You know the length of the wire, the formulas for the area and perimeter of the square, and the formulas for the area and circumference of the circle.

Since the perimeter plus the circumference equals the total length of the wire, you can relate the area to the length of the wire by expressing the perimeter (or sides of the square) in terms of the circumference. This will enable you to express the area in terms of a single variable, namely *c*.

Differentiate the area to find the minimum candidate. Do not forget to check that the candidate is indeed a minimum.

Knowing the circumference tells you where to cut the wire.



Circles pack area more efficiently than squares. So if you want to minimize the sum of the two areas, it makes sense that you would want the square to be bigger.