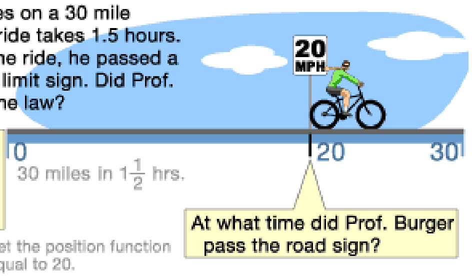
Finding Rate of Change over an Interval

- The **position function** relates time and location. Setting the position function equal to a distance indicates the time an object is at that location.
- When solving a difficult problem, it is a good idea to approximate the answer first. Approximating can give insight into how to work the problem.

Some essential information

Prof. Burger goes on a 30 mile bike ride. The ride takes 1.5 hours. 20 miles into the ride, he passed a 20 mph speed limit sign. Did Prof. Burger break the law?



Prof. Burger's position function: $P(t) = \dfrac{40}{3}t^2$

$P(t) = 20$ Set the position function equal to 20.

$\dfrac{40}{3}t^2 = 20$

$t^2 = 20 \cdot \dfrac{3}{40} = \cancel{20}^{1} \cdot \dfrac{3}{\cancel{40}_{2}}$

$= \dfrac{3}{2}$

$t = \pm\sqrt{\dfrac{3}{2}}$

$t = \sqrt{\dfrac{3}{2}} = \boxed{1.2247...}$ Negative time does not make sense for this problem.

✓ To find the time Prof. Burger is at a given position, set $P(t)$ equal to that distance and solve for t.

In order to analyze the rate of change of an object, it is necessary to know certain things about the objects location at different times.

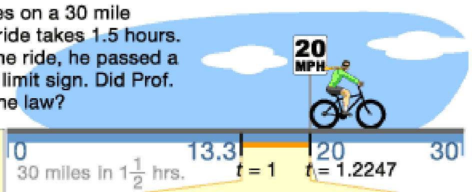
The **position function** is a mathematical relation that connects location to time. By setting time equal to a specific value, the position function will tell you where the object was at that time.

The position function can also be used to determine when an object is at a specific location. By setting the function equal to that distance you can solve the resulting equation for a value of time.

Notice that the situation a problem models can affect which answers are plausible. For example, in most problems time cannot be negative.

Average rate on an interval

Prof. Burger goes on a 30 mile bike ride. The ride takes 1.5 hours. 20 miles into the ride, he passed a 20 mph speed limit sign. Did Prof. Burger break the law?



Prof. Burger's position function: $P(t) = \dfrac{40}{3}t^2$

Consider the interval from $t = 1$ to $t = \sqrt{\dfrac{3}{2}}$.

Find change in time: $\Delta t = 1.2247 - 1$

How do you find the average rate on this part of the bike ride? $R = \dfrac{\Delta P}{\Delta t}$

$P\left(\sqrt{\dfrac{3}{2}}\right) = 20$

$P(1) = \dfrac{40}{3}(1)^2 = \dfrac{40}{3} = 13.333...$

Find change in position: $\Delta P = P\left(\sqrt{\dfrac{3}{2}}\right) - P(1)$

$= \dfrac{P\left(\sqrt{3/2}\right) - P(1)}{\sqrt{3/2} - 1}$

$= \dfrac{20 - 13.333}{1.2247 - 1}$

$= \boxed{29.663...}$

We have already determined that it is not possible to find **instantaneous rate of change** strictly through algebra because of the 0/0 issue.

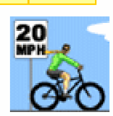
However, approximating an answer often leads to insight. Instead of finding the instantaneous rate of change, try finding the **average rate of change** over a small piece of the bike ride.

Recall that Professor Burger's average rate of change over the entire bike ride was 20 miles per hour. However, his average rate of change over this smaller interval is almost 30 miles per hour. By shrinking the interval it is possible to refine the approximation even further.

Approximate Professor Burger's instantaneous rate of change at $t = \sqrt{\dfrac{3}{2}} \approx 1.2$ hours using the table for his position at time t.

t (hours)	0.0	0.3	0.6	0.9	1.2	1.5	1.8
$P(t)$ (miles)	0.0	1.2	4.8	10.8	19.2	30.0	43.2

$R \approx \dfrac{\Delta P}{\Delta t} = \dfrac{30.0 - 10.8}{1.5 - 0.9} = \dfrac{19.2}{0.6} = \boxed{32 \text{ miles per hour}}$



If the function for Professor Burger's position is not known, but a table of values is, his rate of change can still be approximated. The best approximation to his instantaneous rate of change is the slope of the secant joining the points to the left and to the right of 1.2 hours.

Based on the table, Professor Burger's approximate rate of change when he passed the speed limit sign was 32 miles per hour. He was definitely breaking the law!