Unit: Applications of Integration **Module:** Finding the Area Between Two Curves

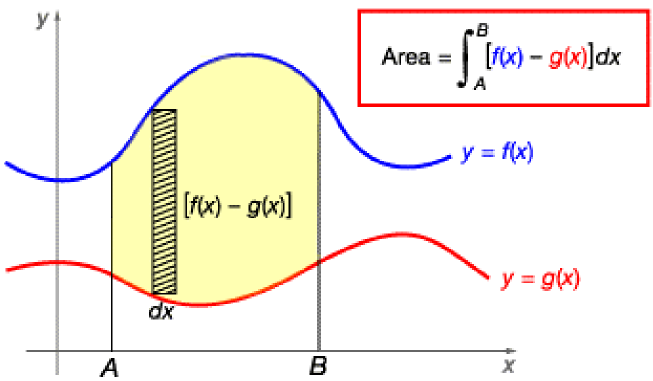
Area Between Two Curves

key concepts:

- The definite integral can be used to calculate the area between a curve and the x-axis on a given interval.
- To find the area of a region bounded by the graphs of two functions, take the definite integral of the difference of the two functions on the appropriate interval.

$$\text{Area} = \int_A^B [f(x) - g(x)]\,dx$$

Definite integrals and area



The **fundamental theorem of calculus** states that if a function f is continuous on an interval $[A,B]$ and F is an antiderivative of f on that interval, then $\int_A^B f(x)dx = F(B) - F(A)$.

Notice that the area between these two regions is equal to the area underneath the top curve minus the area underneath the bottom curve. Since the sum of two integrals is equal to the integral of the sum, you can combine the two integrals into one.

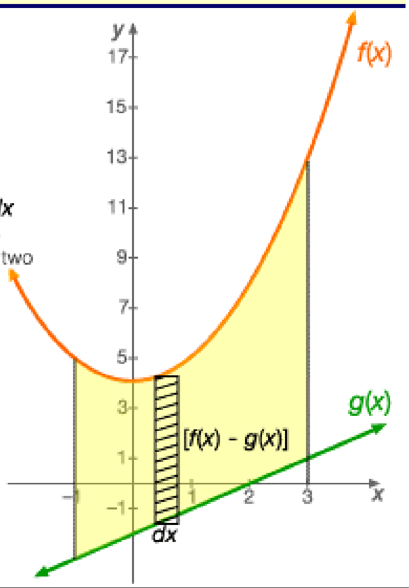
Once you set up the integral, you can use the **fundamental theorem of calculus** to evaluate it.

An area example

Find the area between the curves described by $f(x) = x^2 + 4$ and $g(x) = x - 2$ between the points $x = -1$ and $x = 3$.

$$\text{Area} = \int_{-1}^{3} [(x^2 + 4) - (x - 2)]\,dx$$

The height is the difference of the two functions.



When setting up the integral, remember to take the equation for the upper curve and subtract the equation for the lower curve.

The limits of integration are the x-values where the region begins and ends.

Notice that the integral itself is very basic. Once simplified, it is just the integral of a polynomial equation. You can evaluate the integral piece by piece.

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Limits of Integration and Area

key concepts:

- To find the area of a region bounded by the graphs of two functions, find the limits of integration by determining where the graphs intersect. Then take the definite integral of the difference of the two functions along that interval.

$$\text{Area} = \int_A^B [f(x) - g(x)]\,dx$$

Bounded regions

Find the area of the region bounded by the graphs of $y = x^2$ and $y = 8 - x^2$.

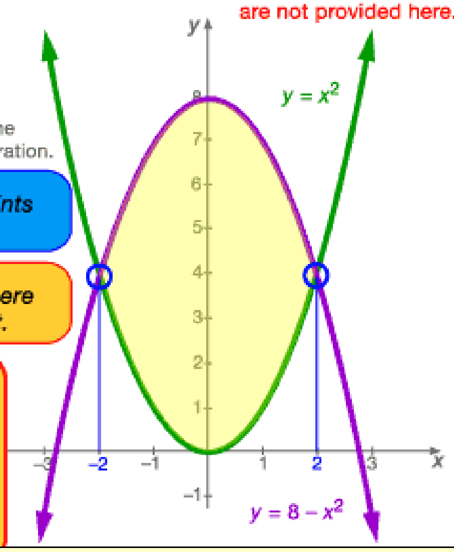
$\text{Area} = \int_{-2}^{2} bh$ Substitute the limits of integration.

Note: The endpoints are not provided here.

Q: *Where are the endpoints for this region?*

A: *The endpoints are where the curves intersect.*

$x^2 = 8 - x^2$
$2x^2 = 8$
$x^2 = 4$
$x = \pm 2$



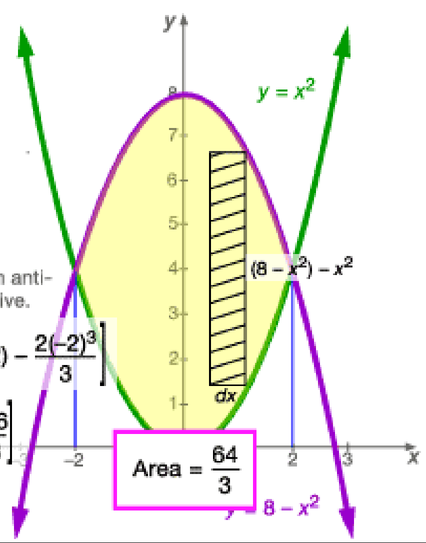
You will have to do some algebra if the endpoints of the region are not stated.

Remember that the two curves cross when the two functions are equal. So set their expressions equal to each other and solve for the endpoints.

Bounded regions

Find the area of the region bounded by the graphs of $y = x^2$ and $y = 8 - x^2$.

$$\text{Area} = \int_{-2}^{2} [(8 - x^2) - x^2]\,dx$$
$$= \int_{-2}^{2} (8 - 2x^2)\,dx$$
$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$$ Find an anti-derivative.
$$= \left[8(2) - \frac{2(2)^3}{3}\right] - \left[8(-2) - \frac{2(-2)^3}{3}\right]$$
$$= \left[16 - \frac{16}{3}\right] - \left[-16 + \frac{16}{3}\right]$$
$$= 32 - \frac{32}{3}$$



Once you have the endpoints, you can set up the definite integral normally.

Remember to take the upper curve and subtract the lower curve.

Be careful! It is very easy to make an algebra or arithmetic mistake while evaluating definite integrals. Always double check that your answer agrees with your sketch of the region.

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Common Mistakes to Avoid When Finding Areas

key concepts:

- When evaluating areas of regions, make sure that the curves do not cross within the corresponding open interval. If they do, it is necessary to evaluate the area of each region separately.
- When finding the area between the x-axis and the curve of a function underneath the axis, multiply the function by –1 to avoid getting a negative area.

Another broken integral

The right way:

Find the area between the x-axis and the graph of $y = 2x + 4$ from $x = -5$ to $x = 1$.

$$\text{Area}_1 = \int_{-2}^{1} [(2x+4) - 0]\,dx$$

$$= \int_{-2}^{1} (2x+4)\,dx$$

$$= \left(x^2 + 4x\right)\Big|_{-2}^{1}$$

$$= [(1)^2 + 4(1)] - [(-2)^2 + 4(-2)]$$

$$= 1 + 4 - 4 + 8$$

$$= 9$$

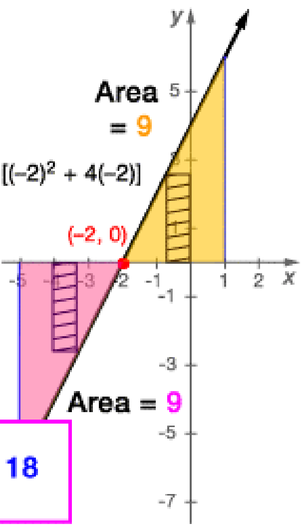
$$\text{Area}_2 = \int_{-5}^{-2} [0 - (2x+4)]\,dx$$

$$= \int_{-5}^{-2} (-2x-4)\,dx$$

$$= \left(-x^2 - 4x\right)\Big|_{-5}^{-2}$$

$$= [-(-2)^2 - 4(-2)] - [-(-5)^2 - 4(-5)]$$

$$= -4 + 8 - (-25 + 20) = 9$$



$\text{Area}_{\text{total}} = 18$

If the curves intersect, it is sometimes necessary to break the integral into two parts.

Anytime the way the arbitrary rectangle's dimensions are defined changes you will have to break the problem up into another piece.

Sometimes you can use the fact that the regions are similar to reduce the extra work you have to do.

The negative area mistake

The right way:

Find the area between the x-axis and the graph of $y = x^2 - 16$ from $x = 0$ to $x = 2$.

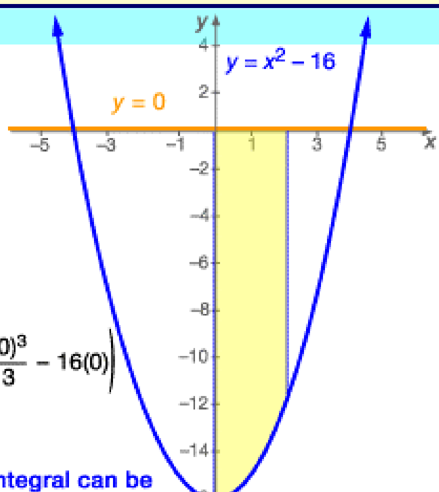
$$\text{Area} = \int_{0}^{2} -(x^2 - 16)\,dx$$

$$= -\left(\frac{x^3}{3} - 16x\right)\Big|_{0}^{2}$$

$$= -\left(\frac{(2)^3}{3} - 16(2)\right) + \left(\frac{(0)^3}{3} - 16(0)\right)$$

$$= -\frac{8}{3} + 32 + 0$$

$$= \frac{88}{3}$$



A definite integral can be negative. But when looking for areas, you must set up a definite integral that will be positive.

A common mistake in finding areas is to assume that the area is equal to the definite integral and work without a picture.

The formula for finding area requires the way you define the arbitrary rectangles to be positive. Area cannot be negative. To avoid this mistake, always consider how the arbitrary rectangle is defined. Notice that finding the area underneath the x-axis but above a curve requires that you put a negative sign into the definite integral.

Definite integrals can be negative. But area cannot. If you get a negative answer when finding areas, check how you defined the dimensions. You might have made an algebra mistake, but it is more likely that you did not consider the fact that the area was below the x-axis.

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Regions Bound by Several Curves

key concepts:

- Sometimes regions can be defined by more than two curves. When finding the area of a region bound by more than two curves, you must break the integral into different pieces wherever the curves bounding the region switch.

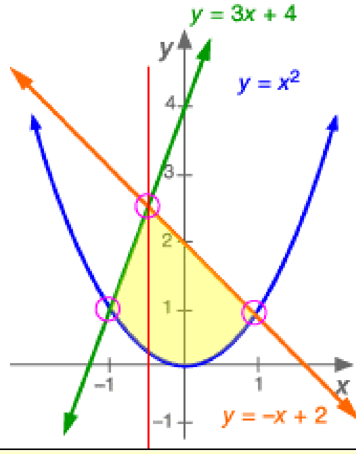
Finding the limits of integration

Find the area bound by the graphs of $y = 3x + 4$, $y = x^2$, and $y = -x + 2$.

$$x^2 = -x + 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } \boxed{x = 1}$$

$$x^2 = 3x + 4$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x = 4 \text{ or } \boxed{x = -1}$$

$$3x + 4 = -x + 2$$
$$4x = -2$$
$$\boxed{x = -\frac{1}{2}}$$



Regions can be defined in many different ways. Some regions are defined by several different curves. Always sketch the region you are working with to identify how the region is defined.

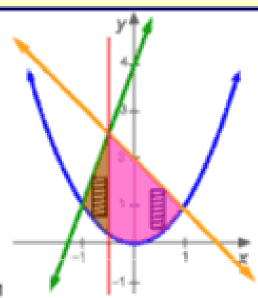
Notice that this region is defined by three curves. Also, the way the dimensions of the arbitrary rectangles are defined changes when the curves intersect.

You will need to work two integrals to find this area.

Setting up the integrals

Find the area bound by the graphs of $y = 3x + 4$, $y = x^2$, and $y = -x + 2$.

$$\text{Area}_{total} = \text{Area}_1 + \text{Area}_2$$
$$= \frac{7}{12} + \frac{27}{12} = \frac{34}{12}$$
$$\text{Area}_{total} = \frac{17}{6}$$



$$\text{Area}_1 = \int_{-1}^{-1/2} (3x + 4 - x^2)\,dx$$
$$= \left[\frac{3x^2}{2} + 4x - \frac{x^3}{3}\right]_{-1}^{-1/2}$$
$$= \left[\frac{3(-1/2)^2}{2} + 4\left(-\frac{1}{2}\right) - \frac{(-1/2)^3}{3}\right] - \left[\frac{3(-1)^2}{2} + 4(-1) - \frac{(-1)^3}{3}\right]$$
$$= \left[\frac{3}{8} - 2 + \frac{1}{24}\right] - \left[\frac{3}{2} - 4 + \frac{1}{3}\right]$$
$$= \left[-\frac{19}{12}\right] - \left[-\frac{26}{12}\right] \boxed{= \frac{7}{12}}$$

$$\text{Area}_2 = \int_{-1/2}^{1} (-x + 2 - x^2)\,dx$$
$$= \left[-\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1/2}^{1}$$
$$= \left[-\frac{(1)^2}{2} + 2(1) - \frac{(1)^3}{3}\right] - \left[-\frac{(-1/2)^2}{2} + 2\left(-\frac{1}{2}\right) - \frac{(-1/2)^3}{3}\right]$$
$$= \left[-\frac{1}{2} + 2 - \frac{1}{3}\right] - \left[-\frac{1}{8} - 1 + \frac{1}{24}\right]$$
$$= \left[\frac{14}{12}\right] - \left[-\frac{13}{12}\right] \boxed{= \frac{27}{12}}$$

The area of the region is equal to the area of the first part of the region plus the area of the second.

Notice that the definite integral changes for the two different pieces. Always take the upper curve and subtract the lower curve. If the curves change, then you will have to take another integral.

Always check that your answer matches the sketch of the region. For example, if you got an answer greater than 4 you would know that you made a mistake somewhere.