Calculus Lecture Notes

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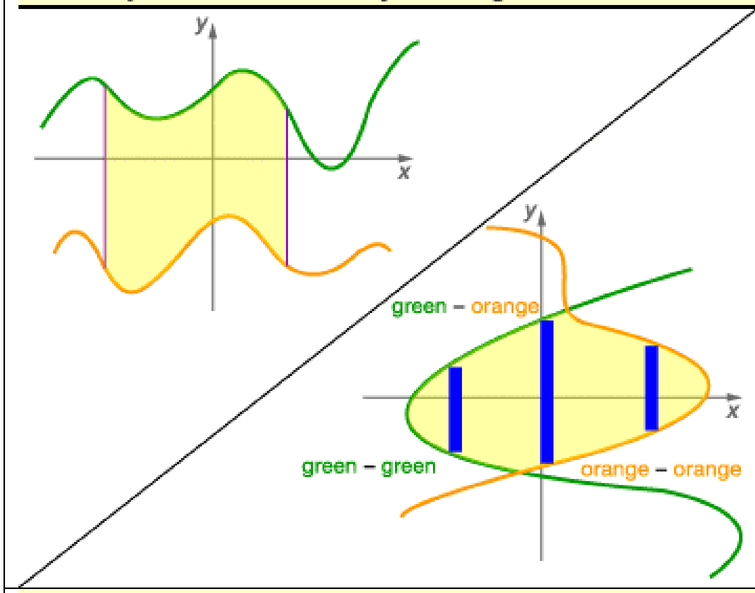
Unit: Applications of Integration **Module:** Integrating with Respect to *y*

Finding Areas by Integrating with Respect to *y* — Part I

key concepts:

- Sometimes it is easier to stack horizontal rectangles instead of vertical rectangles when finding the area bounded by two curves.
- Area between two curves: $\int_A^B [f(x) - g(x)]\, dx$.

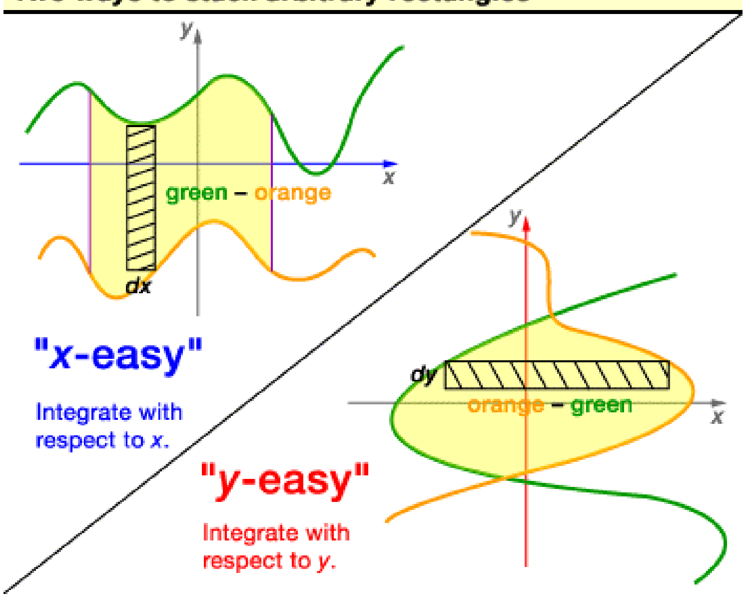
Two ways to stack arbitrary rectangles



You have already seen how to find the area between two curves by integrating with respect to *x*. Integrating with respect to *x* stacks an infinite number of vertical rectangles side by side and adds together their areas.

But sometimes curves are not well defined to integrate with respect to *x*. Notice that you have to change how you define the height of the rectangles bound by these curves. Watch for cases where the curves are not functions of *x*.

Two ways to stack arbitrary rectangles



If the heights of the rectangles are well defined using vertical rectangles, integrate with respect to *x*.

You can stack horizontal rectangles as well. If the heights of the rectangles are well defined using horizontal rectangles, integrate with respect to *y*.

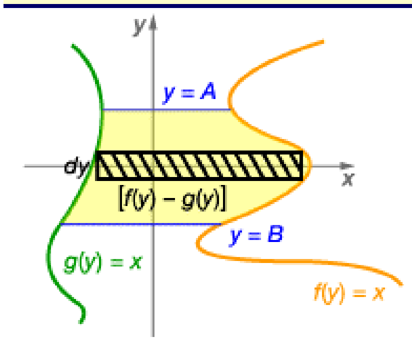
Unit: Applications of Integration **Module:** Integrating with Respect to *y*

Finding Areas by Integrating with Respect to *y* — Part II

key concepts:

- Area between two curves defined in terms of *y*: $\int_B^A [f(y) - g(y)]\, dy$.
- When finding the area of a region:
  1. Sketch the region.
  2. Determine how the rectangles will stack.
  3. Find where the curves intersect.
  4. Set up the integral.
  5. Evaluate.

Area between curves defined in terms of y



$$\text{Area} = \int_B^A [f(y) - g(y)]\, dy$$

"y-easy" "x-easy"

$$\text{Area} = \int_{\text{bottom}}^{\text{top}} [(\text{right}) - (\text{left})]\,(\text{height})$$

$$\text{Area} = \int_{\text{left}}^{\text{right}} [(\text{top}) - (\text{bottom})]\,(\text{base})$$

To integrate with respect to *y*, the expression for the dimensions of the arbitrary rectangles must also be in terms of *y*.

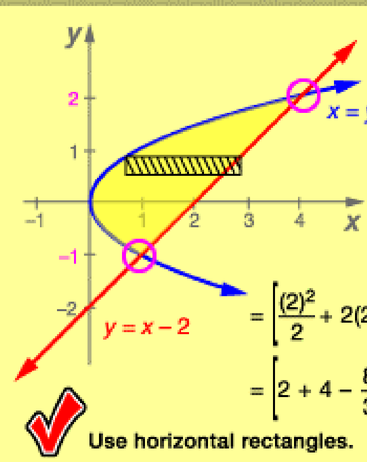
Notice that the height of the rectangle is equal to a small change in *y*. The base is given by the difference in the two *x*-values, which are expressed in terms of *y*.

Notice that the formulas for area use the same basic idea, integrating the area of a rectangle. The dimensions depend on how you define the rectangle.

A "y-easy" problem

What is the area of the region bounded by the graphs of $y = x - 2$ and $x = y^2$?

Sketch th Decide how Find the lim Set up the Evaluate



$$\text{Area} = \int_{\text{bottom}}^{\text{top}} (\text{base})(\text{height})$$

$$\text{Area} = \int_{-1}^{2} [(y + 2) - (y^2)]\, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right]$$

$$= \left[2 + 4 - \frac{8}{3}\right] - \left[\frac{1}{2} - 2 + \frac{1}{3}\right] = \frac{27}{6} = \boxed{\frac{9}{2}}$$

Use horizontal rectangles.

Here is an example.

Start by sketching the graphs.

If you try to stack vertical rectangles, you will have rectangles that are defined by the difference of the same curve. Instead, use horizontal rectangles.

Once you decide how you are going to stack, you must find where the stacking begins and ends. Look for the smallest value used for the variable of integration. That's your lower limit of integration. The greatest value is the upper limit of integration.

Now set up the integral. The height is a small change in *y*. The base is the difference of the two expressions.

Setting up the integral is the trickiest part of this problem. Actually solving the integral is very straightforward.

Unit: Applications of Integration **Module:** Integrating with Respect to *y*

Area, Integration by Substitution, and Trigonometry

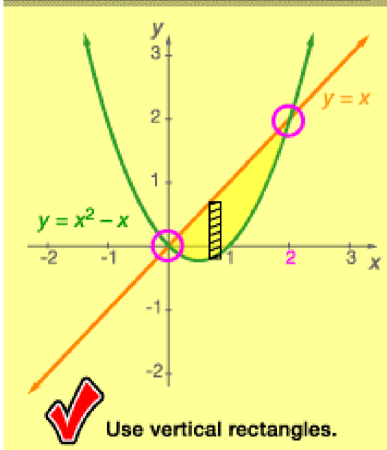
key concepts:

- Area between two curves: $\int_A^B [f(x) - g(x)]\, dx$.
- When finding the area of a region:
  1. Sketch the region.
  2. Determine how the rectangles will stack.
  3. Find where the curves intersect.
  4. Set up the integral.
  5. Evaluate.

Deciding how to stack rectangles

What is the area of the region bounded by the graphs of $y = x$ and $y = x^2 - x$?

Sketch th | Decide how | Find the lim | Set up the | Evaluate



$$\text{Area} = \int_0^2 [x - (x^2 - x)]\, dx$$

$$\text{Area} = \int_0^2 (-x^2 + 2x)\, dx$$

$$= \left[-\frac{x^3}{3} + x^2\right]_0^2$$

$$= \left[-\frac{(2)^3}{3} + 2^2\right] - \left[-\frac{(0)^3}{3} + (0)^2\right]$$

$$= -\frac{8}{3} + 4 - 0 \quad = \boxed{\frac{4}{3}}$$

The best tip for working area problems is to pay special attention to the way you choose to stack the rectangles.

In this example, the curves are defined in terms of *x*. Vertical rectangles stack well for these curves. If the curves were defined in terms of *y*, horizontal rectangles would probably stack better.

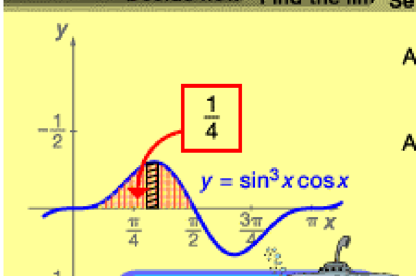
Make sure that your limits of integration correspond to the variable of integration. If you are integrating with respect to *x*, then the limits of integration will be *x*-values.

Always draw in an arbitrary rectangle. Use that rectangle to determine the dimensions you will use to integrate.

A trigonometric example

What is the area of the region bounded by the graph of $y = \sin^3 x \cos x$ and the *x*-axis between $x = 0$ and $x = \pi/2$?

Sketch th | Decide how | Find the lim | Set up the | Evaluate



Let $u = \sin x$.
$du = \cos x\, dx$

$$\int u^3\, du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

$$\text{Area} = \int_{\text{left}}^{\text{right}} (\text{height})(\text{base})$$

$$\text{Area} = \int_0^{\pi/2} \sin^3 x \cos x\, dx$$

$$= \int_0^{\pi/2} (\sin x)^3 \cos x\, dx$$

$$= \left[\frac{\sin^4 x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4} - 0 = \boxed{\frac{1}{4}}$$

Once you get the hang of setting up the integral, you will be able to compute the area of very exotic shapes.

Keep in mind the different techniques of integration when evaluating the definite integral. Here the integral can be solved with a simple substitution.

Remember to convert back to the original variable before you evaluate the integral along the interval.

Notice that the integration by substitution is performed on an indefinite integral. To work the problem without this step you would have to express the limits of integration in terms of *u*.