Calculus Lecture Notes

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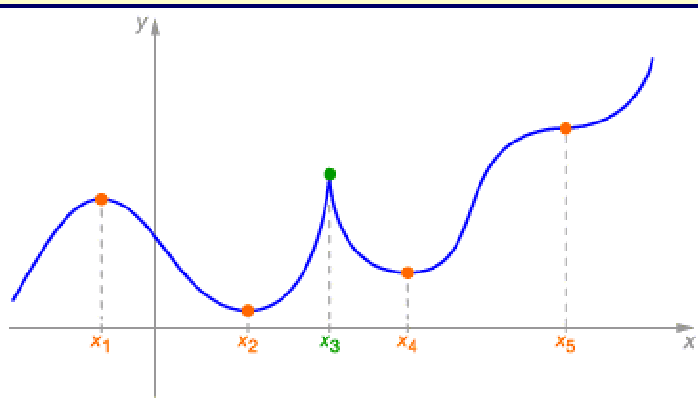
Unit: Curve Sketching **Module:** Critical Points

Critical Points

key concepts:

- **Critical points** are the places on a graph where the derivative equals zero or is undefined. Interesting things happen at critical points.
- To find critical points, take the derivative, set the derivative equal to zero and solve, and find values where the derivative is undefined.

Looking for interesting points



Consider this graph of some complicated function. There seem to be several interesting points.

At x_1, x_2, x_4, and x_5, the tangent lines are horizontal, so the derivative of the function is 0 at each of those values.

At x_3, the tangent line is vertical. Its slope is undefined, so the derivative of the function at that point is undefined, too.

These points are called **critical points**. To identify them you need to solve for values that make the derivative equal to zero or undefined.

example: Find the **critical points** of $f(x) = 4x^2 + 2x - 2$.

Here is an example of a function that you might want to graph.

Find the derivative $f'(x) = 8x + 2$

To find its critical points, first take the derivative.

Solve $f'(x) = 0$ $8x + 2 = 0$

$\boxed{x = -\frac{1}{4}}$ **critical point**

Next set the derivative equal to zero and solve.

~~**Find where $f'(x)$ is undefined**~~ Since the derivative is a linear function, it is defined for all values of x.

Finally, look for places where the derivative is undefined. In this case, the derivative is defined for all real numbers.

example: Find the **critical points** of $h(x) = x^{1/3}$.

You will need to use the power rule to take the derivative of this function.

Find the derivative $h'(x) = \frac{1}{3}x^{-2/3}$

$= \frac{1}{3x^{2/3}}$

~~**Solve $h'(x) = 0$**~~ The derivative is never equal to 0 since the numerator is always 1.

First, notice that there are no values of x that make the derivative equal to zero.

Find where $h'(x)$ is undefined $3x^{2/3} = 0$

$\boxed{x = 0}$ **cusp critical point**

The derivative is undefined when its denominator equals zero.

Next, look for values of x for which the derivative is undefined. One way to do this is to set the denominator equal to zero and solve. In this case, the derivative is undefined for $x = 0$, even though the function itself is defined at that point. Therefore you may have a cusp point at that x-value.

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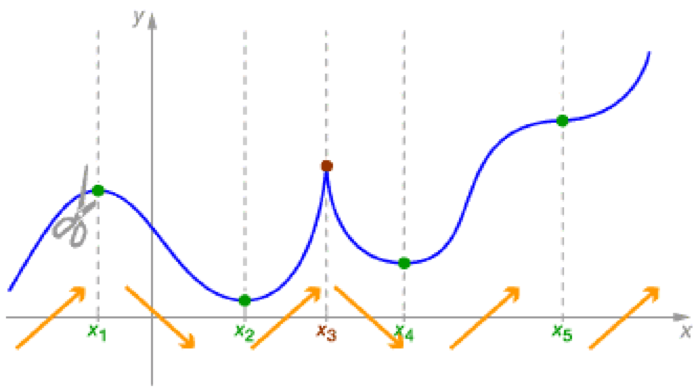
Regions Where a Function Increases or Decreases

key concepts:

- **Critical points** divide the curve of a function into regions where the function either **increases** or **decreases**.
- On an interval, the sign of the first derivative of a function indicates whether that function is increasing or decreasing.

A review of critical points



Consider the regions of this graph that are defined by the critical points.

Between any two adjacent **critical points** the curve behaves in only one way. Either it is **increasing** or it is **decreasing**.

Notice that there is no way for the behavior of the curve to change without it either leveling off or making a sharp turn. When either situation occurs, there is a critical point, so they indicate a possible change from increasing to decreasing or decreasing to increasing.

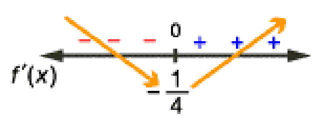
However, a critical point does not always force a curve to change its behavior. For example, x_5 is a critical point, but the function is increasing on both the right and the left.

A quadratic function

example: $f(x) = 4x^2 + 2x - 2$

$f'(x) = 8x + 2$ derivative

critical point at $x = -\frac{1}{4}$



Let $x = -1$. Pick an easy point to the left.

$f'(-1) = 8(-1) + 2$
$= -8 + 2$
$= -6$ negative

All values in this region are negative.

Let $x = 0$. Pick an easy point to the right.

$f'(0) = 8(0) + 2$
$= 0 + 2$
$= 2$ positive

All values in this region are positive.

f is decreasing for all $x < -\frac{1}{4}$.

f is increasing for all $x > -\frac{1}{4}$.

f has a minimum value at $x = -\frac{1}{4}$.

Here is an example of a function whose critical point you have already determined.

To study the behavior of the curve, make a sign chart. To keep it simple, just label the critical point.

Next choose any x-value that is less than the critical value. Here an easy value is $x = -1$. Since the derivative at $x = -1$ is negative, all the values to the left of the critical point will make the derivative negative. The function is decreasing on that region. Mark a down-arrow on the sign chart.

You must also choose an x-value to the right of the critical point. Here an easy value is $x = 0$. Since that value makes the derivative positive, the function is increasing on that region. Mark an up-arrow on the sign chart.

Since the function is decreasing to the left of the critical point and increasing to the right, you can conclude that the curve reaches a minimum point at the critical point.

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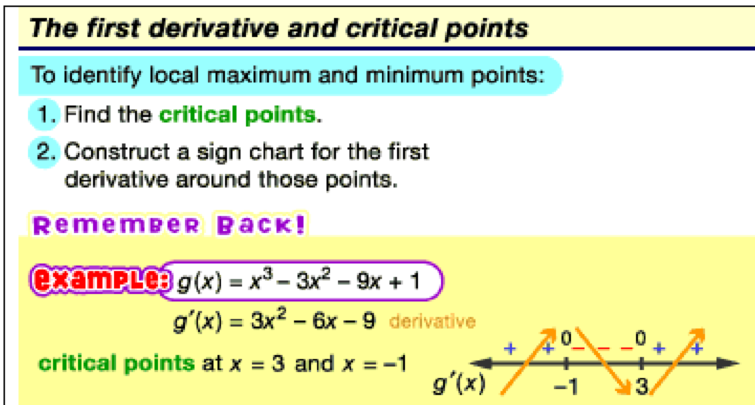
The First Derivative Test

key concepts:

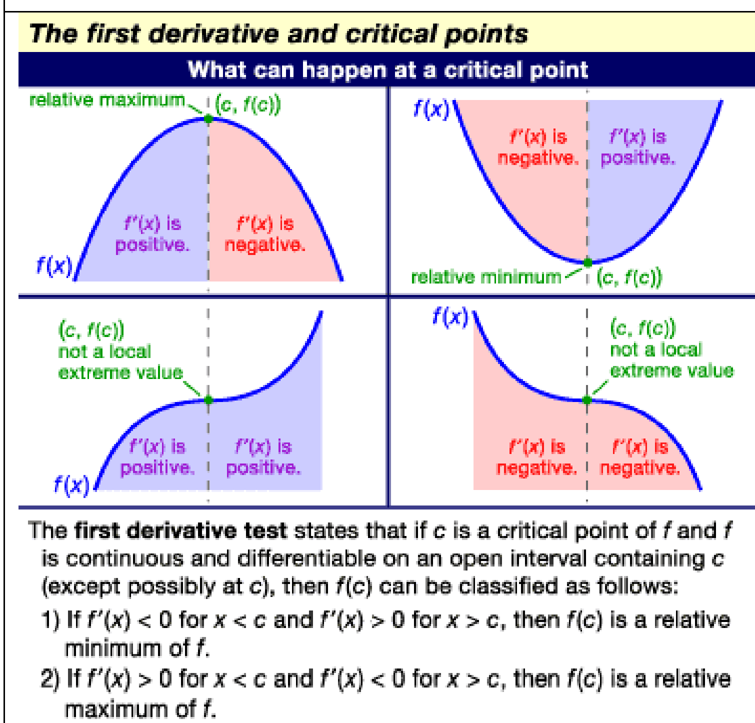
- Determining the sign of the derivative immediately to either side of a **critical point** reveals whether that point is a **relative maximum**, **relative minimum**, or neither.
- The **first derivative test** states that if c is a critical point of f and f is continuous an differentiable on an open interval containing c (except possibly at c), then $f(c)$ can be classified as follows:
  1) If $f'(c) < 0$ for $x < c$ and $f'(c) > 0$ for $x > c$, then $f(c)$ is a relative minimum of f.
  2) If $f'(c) > 0$ for $x < c$ and $f'(c) < 0$ for $x > c$, then $f(c)$ is a relative maximum of f.
- On an interval, the sign of the first derivative of a function indicates whether that function is increasing or decreasing.



Maximum and minimum values of a function will only occur at **critical points**: places where the derivative is zero or undefined.

Once you have identified the critical points, use a sign chart to illustrate the behavior of the graph on either side of each critical point.

The sign chart for $g'(x)$ shows that the function g is increasing to the left of $x = -1$ and decreasing to the right. Therefore g has a maximum value at $x = -1$ Since the g is decreasing to the left of $x = 3$ and increasing to the right of it, there is a minimum value at $x = 3$.



Take a look at the possible situations around a critical point.

If the function changes from increasing to decreasing, then it will have a **relative maximum** value at the critical point. If the function changes from decreasing to increasing, then it will have a **relative minimum** at the critical point.

If the function does not change behavior on either side of the critical point, then it will have neither a maximum value nor a minimum value at the critical point.

Notice the general relationship between the sign of the derivative, the regions where the function is increasing or decreasing, and the existence of maximum or minimum points. Using the **first derivative test** you can move directly from the sign of the derivative to the existence of maxima and minima.