Derivatives of Exponential Functions

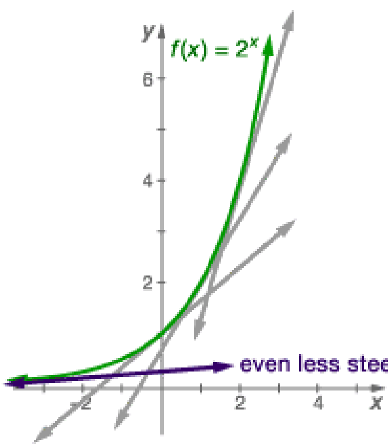
key concepts:

- Studying the slopes of tangent lines to the graph of a function can help you determine the derivative.
- The derivative of an **exponential function** is the product of the function and the **natural log** of its base.

Using the graphs of the exponential functions

Consider $f(x) = 2^x$.

What is the derivative?



remember: The derivative produces the slopes of the tangent lines.

* The derivative will always be positive.

Before trying to find the derivative of an **exponential function**, it is a good idea to examine the behavior of the lines tangent to the function.

Since the base is greater than 1, the tangent lines are always positive. In addition, the slopes of the tangent lines seem to be increasing.

Using the definition of the derivative

Consider $f(x) = N^x$, $N > 0$.

$$f'(x) = \boxed{\lim_{\Delta x \to 0} \frac{N^{x+\Delta x} - N^x}{\Delta x}}$$

indeterminate form

$$= \lim_{\Delta x \to 0} \frac{N^x N^{\Delta x} - N^x}{\Delta x}$$

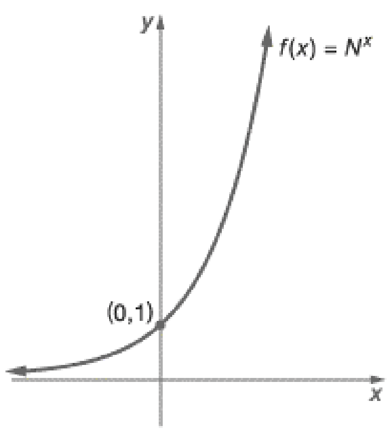
$$= \lim_{\Delta x \to 0} \frac{N^x\left(N^{\Delta x} - 1\right)}{\Delta x}$$

$$= N^x \lim_{\Delta x \to 0} \frac{\left(N^{\Delta x} - 1\right)}{\Delta x}$$

$$= N^x L(N)$$ Call the limit $L(N)$.

$$f'(x) = f(x)L(N)$$

the original function times some number



Use the definition of the derivative to find the derivative of a general exponential function.

Notice that you can factor the exponential function out of the limit. You can do this because there are no Δx-terms in that factor.

The remaining factor that includes the limit portion is harder to evaluate. However, it is apparent that the derivative of the exponential function is equal to that same exponential function times the result of that limit.

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Approximating L(N)

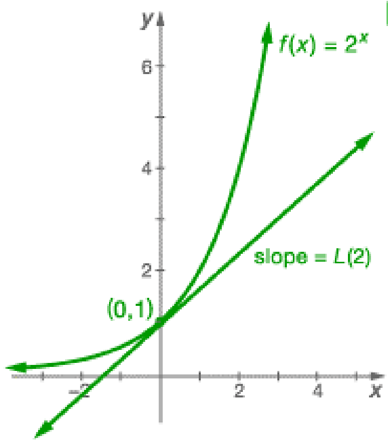
Consider $f(x) = N^x$, $N > 0$.

$f'(x) = f(x)L(N)$ the original function times some number

Evaluate $f'(0)$. $f'(0) = L(N)$

Approximate $L(2)$.

$$L(2) = \lim_{\Delta x \to 0} \frac{(2^{\Delta x} - 1)}{\Delta x}$$



Δx	$\frac{(2^{\Delta x} - 1)}{\Delta x}$
0.1	0.717735
0.01	0.695555
0.001	0.693387
0.0001	0.693171

Even though you cannot solve the limit directly, you can approximate the value of the slope for an exponential function at a point.

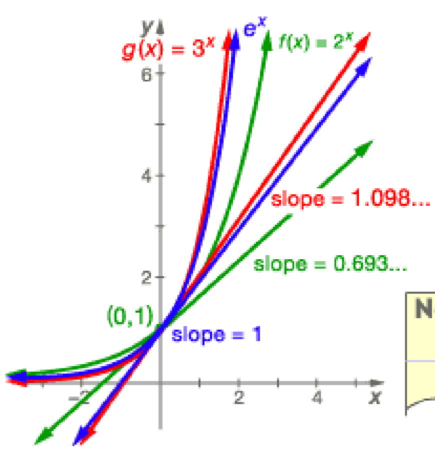
Remember, smaller values of Δx will give you better approximations of the value of the limit.

A special derivative

Consider $f(x) = e^x$.

$$f'(x) = e^x L(e)$$
$$= e^x \qquad L(e) = 1.$$

Amazing!



Note: For $f(x) = N^x$, $N > 0$,
$f'(x) = N^x \ln N$.

It turns out that the limit in question is equal to the **natural log** of the base of the exponential function.

Therefore the derivative of the **natural exponential function** is itself, since the natural log of e is 1.

Here is the formula for the derivative of the general exponential function.