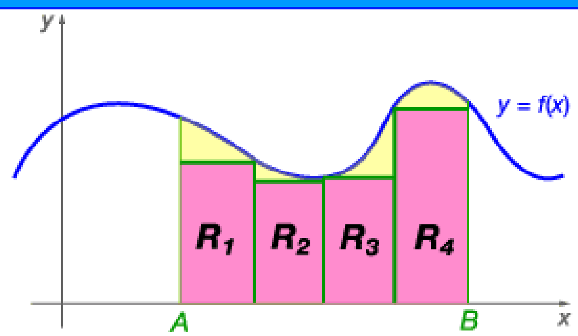
Unit: The Basics of Integration **Module:** The Fundamental Theorem of Calculus

Approximating Areas of Plane Regions

key concepts:

- The two key questions of calculus have a subtle connection.
- When trying to find the area of a complicated region, try approximating the area with rectangles. As the number of rectangles increases, the approximation becomes more accurate.

Q2: *How do you find the area of exotic shapes?*



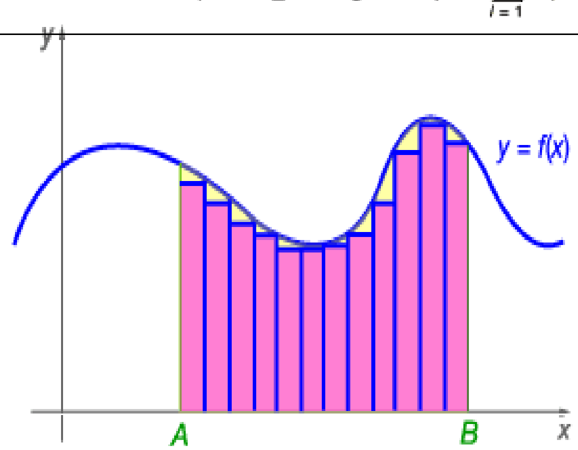
A: *Approximate the area by dividing the region into rectangles.*

$$\text{Area} = R_1 + R_2 + R_3 + R_4 = \sum_{i=1}^{4} R_i$$

The two big questions in calculus are “How do you find instantaneous velocity?” and “How do you find the area of exotic shapes?”

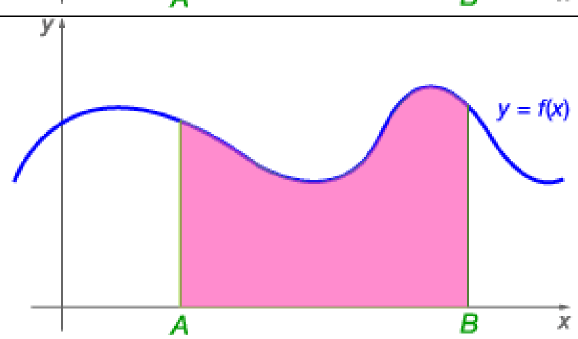
Consider an exotic shape defined by the graph of a function, the *x*-axis and two points on the axis.

Since calculating the area of the region might be difficult, you could approximate the area by dividing the region into rectangles. The area covered by the rectangles can be expressed as the sum of the areas of the individual rectangles. Sigma (Σ) notation provides a shorthand expression for the sum.



You can improve the approximation of the area by increasing the number of rectangles. With more rectangles, less of the region is left uncovered.

Notice that the base of each rectangle is thinner than with fewer rectangles. As the number of rectangles increases, the lengths of the bases will approach zero.



To completely cover the region, you will need infinitely many rectangles whose bases are infinitesimal. The sum of their areas will equal the area of the region.

In other words, you need to take the limit of the areas of the rectangles as the lengths of their bases approach zero.

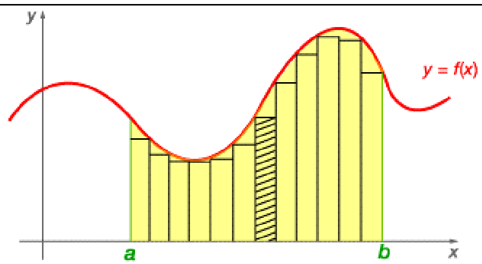
Unit: The Basics of Integration **Module:** The Fundamental Theorem of Calculus

Areas, Riemann Sums, and Definite Integrals

key concepts:

- As the number of rectangles used to approximate the area of region increases, the approximation becomes more accurate. It is possible to find the exact area by letting the width of each rectangle approach zero, thus generating an infinite number of rectangles.
- The **Riemann sum** of f for the partition Δ is the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$

  where $x_{i-1} \le c_i \le x_i$, f is defined on $[a, b]$, Δ is a partition of $[a, b]$ given by $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ and Δx_i is the length of the ith subinterval.
- A function and the equation for the area between its graph and the x-axis are related by the antiderivative.
- The **definite integral** of f from a to b is the limit of the Riemann sum as the lengths of the subintervals approach zero.



$A \approx$ sum of the areas of the rectangles

$= \int_{um}$ (height) • (base) $= \sum_{i=1}^{n}$(height) • (base)

$= \int_{um} f(x)\,\Delta x = \sum_{i=1}^{n} f(c_i)\Delta x_i$

As the bases of the rectangles approach zero, the approximation becomes exact.

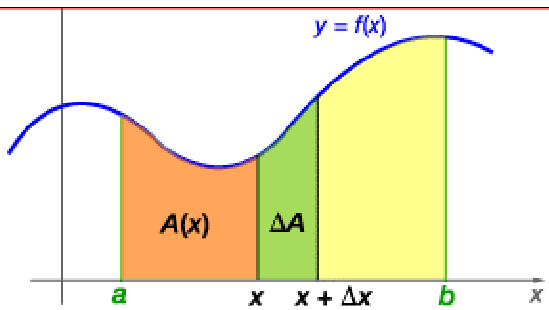
$A = \int_a^b f(x)\,dx$ The formula looks like an integral.

One way to approximate the area of a region is to fill it with rectangles. The sum of their areas will be an approximation for the area of the region.

The area of a rectangle is height times base. You can use sigma (Σ) notation for the sum of the n rectangles, where n is a whole number.

You can represent the height of a given rectangle by $f(x)$ and the base by Δx, a tiny change in the x-direction.

The result resembles an integral.



$\Delta A \approx f(x)\,\Delta x$

$\frac{\Delta A}{\Delta x} \approx f(x)$ Divide by Δx.

$\frac{dA}{dx} = f(x)$ Consider the same relationship as the change in area and base approach zero. The relationship looks like a derivative.

$A = \int f(x)\,dx$

Here is a justification for using an integral to compute area.

Let $A(x)$ be a function that gives the area to the left of x. The next slice of area, ΔA, can be approximated by a rectangle whose height is the value $f(x)$ and whose base is.

Solving for $f(x)$ and letting ΔA and Δx become very small produces the relationship $f(x) = dA/dx$.

Therefore A is the integral of $f(x)$.

Calculus Lecture Notes

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Unit: The Basics of Integration **Module:** The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

key concepts:

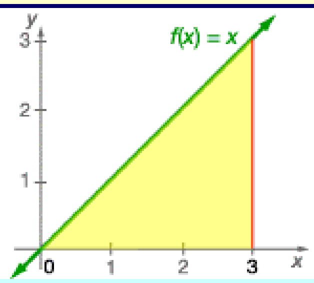
- Let f be defined on the interval $[a,b]$. The **definite integral** of f from a to b is
$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx \text{ if } \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i \text{ exists.}$$
- The **fundamental theorem of calculus** links the velocity and area problems. It enables you to evaluate definite integrals, thereby finding the area between a curve and the x-axis.
- The fundamental theorem of calculus states that if f is continuous on $[a,b]$ and F is an antiderivative of f on that interval, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

A basic example

example:

Consider $f(x) = x$.

Find the area between the graph of f and the x-axis on the interval $[0, 3]$.



Area by algebra:

$$\text{Area} = \frac{bh}{2} = \frac{(3)(3)}{2} = \boxed{\frac{9}{2}}$$

Area by calculus:

$$\int_0^3 x\,dx = \left(\frac{x^2}{2}+C\right)\Bigg|_0^3 = \left(\frac{9}{2}+\cancel{C}\right)-\left(\frac{0}{2}+\cancel{C}\right) = \frac{9}{2}-0 = \boxed{\frac{9}{2}}$$

The C-terms always cancel when dealing with definite integrals.

The **fundamental theorem of calculus** provides a means of evaluating **definite integrals**. These are the integrals that are associated with calculating areas under curves.

A simple example involves calculating the area under the line described by $f(x) = x$ on the interval $[0,3]$. Since the region involved is a triangle, you can use the area formula for triangles to arrive at the answer.

The calculus technique requires you to evaluate the definite integral of f from 0 to 3.

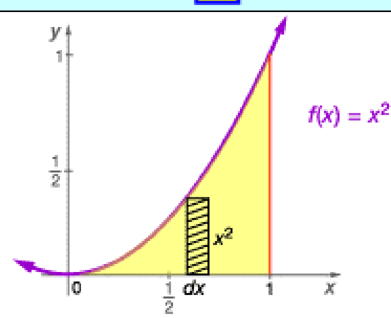
Notice that the constant of integration C cancels with itself.

The definite integral produces the same result as the area formula.

example:

Consider $f(x) = x^2$.

Find the area between the graph of f and the x-axis on the interval $[0, 1]$.



$$\text{Area} = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)\Bigg|_0^1 = \left(\frac{(1)^3}{3}\right)-\left(\frac{(0)^3}{3}\right) = \boxed{\frac{1}{3}}$$

Evaluate the expression at 1 and subtract the value of the expression at 0.

You can use definite integrals to determine areas of more unusual regions. Here you have the area under a parabola.

Set up the definite integral of the function f from 0 to 1 and evaluate it.

The fundamental theorem tells you to evaluate the antiderivative at 1 and subtract the value of the antiderivative at 0.

The area of the region is 1/3.

Unit: The Basics of Integration **Module:** The Fundamental Theorem of Calculus

Illustrating The Fundamental Theorem of Calculus

key concepts:

- The **fundamental theorem of calculus** enables you to evaluate **definite integrals,** thereby finding the area between a curve and the *x*-axis.

$$\text{Area} = \int_a^b f(x)\,dx$$

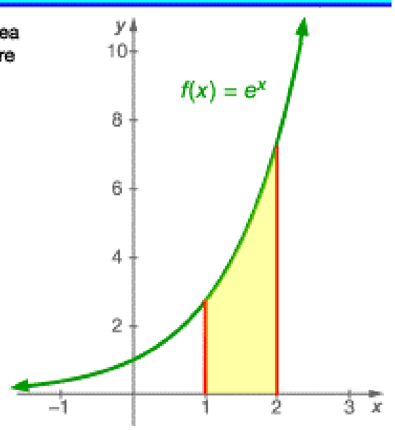
- When the limits of integration are not given by the problem, find them by determining where the curve intersects the *x*-axis.
- The fundamental theorem of calculus states that if *f* is continuous on $[a, b]$ and *F* is an antiderivative of *f* on that interval, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Q: What is the area between the graph of $f(x) = e^x$ and the *x*-axis between $x = 1$ and $x = 2$?

TIP: Draw a picture to get a better idea of what is being asked and to make sure your final answer makes sense.

$$\text{Area} = \int_1^2 e^x\,dx$$
$$= e^x\Big|_1^2$$
$$= e^2 - e^1$$

A: Area $= e(e - 1)$



The **fundamental theorem of calculus** implies that you can calculate the area between a curve and the *x*-axis by finding the antiderivative of the function and evaluating it at the specified endpoints.

You can calculate the area of the region under a curve without graphing it. However, it is a good idea to draw a picture to make sure your answer makes sense.

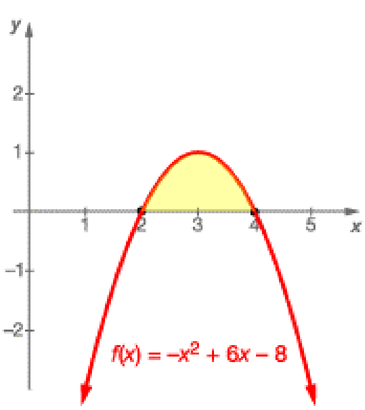
Q: What is the area under the graph of $f(x) = -x^2 + 6x - 8$ that is above the *x*-axis?

Where is the curve above the *x*-axis?

$$f(x) = -x^2 + 6x - 8$$

parabola

sad-faced



Find the intercepts:

$$-x^2 + 6x - 8 = 0$$
$$x^2 - 6x + 8 = 0$$

Multiply by –1.

$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

Sometimes the endpoints may not be specified from the outset.

You want the area of the region below the curve and above the *x*-axis. The function is a quadratic, so the curve will be parabolic. The negative coefficient of *x*-squared means that the parabola will open downward.

First determine if the curve crosses the *x*-axis. The equation of the *x*-axis is $y = 0$, so set the function equal to 0 and solve for *x*.

There are two solutions, which correspond to the *x*-intercepts of the parabola. They are the endpoints of the region. Use them as the limits of integration of the definite integral of $f(x)$ to determine the area of the region.

Calculus Lecture Notes

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Unit: The Basics of Integration **Module:** The Fundamental Theorem of Calculus

Evaluating Definite Integrals

key concepts:

- When working with integration by substitution and **definite integrals**, the limits of integration are given in terms of the original variable.
- Since there is a connection between the definite integral and the area between a curve and the x-axis, some definite integrals can be solved geometrically.

Working on definite integrals

Evaluate $\int_0^1 x\sqrt{x^2+1}\,dx$.

u-SUB

Let $u = x^2 + 1$.

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int\sqrt{x^2+1}\,du$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$$

The arbitrary constant will not be needed when you evaluate the definite integral.

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

Professor Burger finds the antiderivative by considering the indefinite integral. Otherwise he would have to convert the limits of integration if he set the expressions equal to one another.

Evaluate $\int_0^1 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[(x^2+1)^{3/2}\right]_0^1$

$$= \boxed{\frac{2\sqrt{2}-1}{3}}$$

Definite integrals appear with limits of integration. They produce numerical values for as results. Geometrically a definite integral represents the area between the curve described by the integrand and the x-axis.

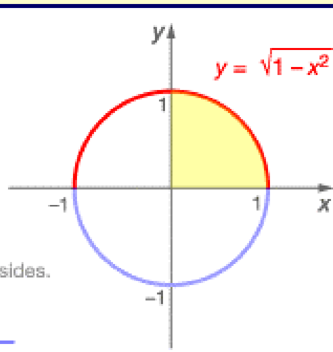
When you evaluate definite integrals by substitution, the limits of integration are x-values, not u-values. One way to avoid this difficulty is to determine the antiderivative using an indefinite integral.

Once you have determined the antiderivative, you can evaluate the indefinite integral. You do not need the constant of integration C, since it will be canceled.

A very tricky integral

Evaluate $\int_0^1 \sqrt{1-x^2}\,dx = \boxed{\frac{\pi}{4}}$

Try looking at the problem geometrically instead of algebraically.



$$y = \sqrt{1-x^2}$$

$$y^2 = \left(\sqrt{1-x^2}\right)^2 \quad \text{Square both sides.}$$

$$x^2 + y^2 = 1$$

$$A_O = \pi r^2$$

$$\frac{A_O}{4} = \frac{\pi r^2}{4}$$

$$A = \frac{\pi}{4}$$

This integrand does not resemble any of the basic patterns, and the choices for integration by substitution do not seem to simplify the integrand.

One way to better understand this integral is to consider it graphically.

By setting y equal to the integrand, you can square both sides and arrive at the equation of a circle. Since the limits of integration are $x = 0$ and $x = 1$, the region is one quarter of a circle of radius 1.

Use the formula for the area of a circle and divide by 4 to arrive at the area of the region. The result is the value of indefinite integral.