Calculus Lecture Notes

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Unit: The Basics of Integration **Module:** Integration by Substitution

Undoing the Chain Rule

key concepts:

- Since differentiation and integration are inverse operations, some of the patterns used when differentiating can be seen when working with integrals.
- One method for evaluating integrals involves untangling the chain rule. This technique is called **integration by substitution**.

Some warm-up problems

Warm-up #1: Find $\frac{dy}{dx}$ if $y = (5x^3 + 2x)^4$.

$y = (5x^3 + 2x)^4$

$\frac{dy}{dx} = 4(5x^3 + 2x)^3 \cdot (15x^2 + 2)$

Here are some warm-up problems.

Remember, to find the derivative of a composite function you must use the chain rule.

Notice that the derivative is the product of a composite function and the derivative of the inside.

Warm-up #2: Find $\frac{dy}{dx}$ if $y = \ln\left[\sin(3x^2)\right]$.

$y = \ln\left[\sin(3x^2)\right]$

$\frac{dy}{dx} = \frac{1}{\sin(3x^2)} \cdot \frac{d}{dx}\left[\sin(3x^2)\right]$

$\frac{dy}{dx} = \frac{1}{\sin(3x^2)} \cdot \cos(3x^2) \cdot \frac{d}{dx}(3x^2)$

$\frac{dy}{dx} = \frac{1}{\sin(3x^2)} \cdot \cos(3x^2) \cdot (6x)$

This derivative is the product of a composite function, another composite function, and the derivative of the inside of the second composite function.

The **chain rule** states if $f(x) = g(h(x))$ where g and h are differentiable functions, then f is differentiable and $f'(x) = g'(h(x)) \cdot h'(x)$.

Warm-up #3: Evaluate $\int 4(15x^2 + 2)(5x^3 + 2x)^3\, dx$

You have already found a function whose derivative is the expression in the integrand, so you already have an antiderivative.

$$\int 4(15x^2 + 2)(5x^3 + 2x)^3\, dx = (5x^3 + 2x)^4 + C$$

If the derivative of the inside is sitting elsewhere in the integrand, then you can use a technique called integration by substitution to evaluate the integral.

Here is a trick question. You could solve this indefinite integral by multiplying everything out and working it term by term. However, there is an easier way.

Notice that the integrand is equal to one of the derivatives you found above. So you already know a function that produces this integrand as its derivative. Since that is what integration finds, that means you already know the integral.

When you see a composite function multiplied by its derivative in the integrand, it is a good hint that you can use a technique to evaluate the integral called **integration by substitution**.

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Unit: The Basics of Integration **Module:** Integration by Substitution

Integrating Polynomials by Substitution

key concepts:

- The differential of an integral identifies the variable of integration.
- **Integration by substitution** is a technique for finding the antiderivative of a composite function. To integrate by substitution, select an expression for *u*. Next, rewrite the integral in terms of *u*. Then, simplify the integral and evaluate.

A little about notation

this...	means this
$\frac{d}{dx}$	differentiate with respect to *x*
$\int dx$	integrate with respect to *x*

When using Leibniz notation, the expression underneath the bar indicates the variable with respect to which the derivative is taken.

The same expression appears when working with integrals. Integrate with respect to the variable indicated by this expression.

Running the chain rule backwards

Evaluate $\int 42x(x^2+4)^{20}\,dx$.

$$\int 42x(x^2+4)^{20}\,dx \neq \int 42x\,dx \bullet \int (x^2+4)^{20}\,dx$$

The integral of a product is not necessarily equal to the product of the integrals.

$$\int 42x(x^2+4)^{20}\,dx = \int 42x(u)^{20}\,dx$$

Let $u = x^2 + 4$.

$$= \int 42(u)^{20}x\,dx$$

$\frac{du}{dx} = 2x$ Take the derivative of *u*.

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$ Substitute into the integral.

$$= \int 42(u)^{20}\frac{1}{2}\,du$$

$$= \int 21(u)^{20}\,du$$

$$= \frac{\cancel{21}u^{21}}{\cancel{21}} + C$$

$$= u^{21} + C$$

$$\boxed{= (x^2+4)^{21} + C}$$

Note: Always express your answer in terms of the original variables.

Notice that the integrand is the product of a composite function and the derivative of its inside. A composite function is a sign that you should try **integration by substitution**.

As with derivatives, the integral of a product of two functions is not equal to the product of the integrals. Find a way to transform the integral into something you can evaluate.

Here let *u* be the inside of the composite function.

Notice that the derivative of *u* is contained within the integrand.

Substitute so that you remove all of the *x*-terms from the integrand. The resulting integral is one you can evaluate with the power rule. This is how you integrate by substituting.

Do not forget the constant of integration.

When using integration by substitution, always express the answer in terms of the original variable.

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