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Unit: Curve Sketching **Module:** Asymptotes

Vertical Asymptotes

key concepts:

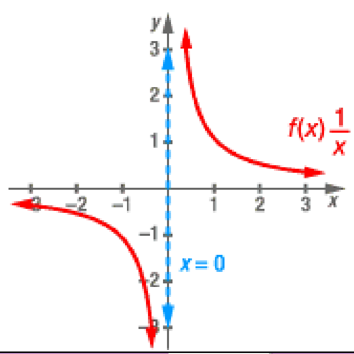
- **Asymptotes** are lines that the graph of a function approaches. A **vertical asymptote** to the graph of a function f is a line whose equation is $x = a$ where $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.
- Identify vertical asymptotes for a **rational function** by factoring the numerator and denominator, canceling where possible, and determining where the resulting denominator is zero.
- If a given point makes both the numerator and denominator of a function equal zero, then there might be a hole in the graph of the function at that point.

The reciprocal function

Consider $f(x) = \frac{1}{x}$.

1ST No factoring is possible.
2ND No canceling is possible.
3RD The denominator is 0 when $x = 0$.

The line given by $x = 0$ is a **vertical asymptote**.



The graphs of some **rational functions** have **vertical asymptotes**.

To determine vertical asymptotes you must follow three steps. First factor the numerator and denominator. Second, cancel any factors they have in common. Third, set the resulting denominator equal to zero and solve for x.

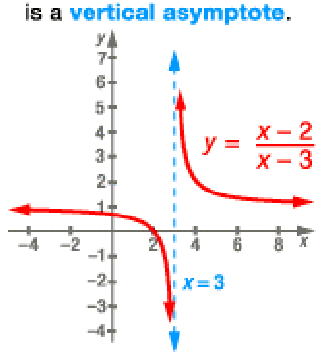
For the reciprocal function, no factoring or canceling is possible. When you set the denominator equal to zero, the result is $x = 0$. This is the equation of the vertical asymptote.

If a function has a vertical asymptote, then its graph will get extremely close to the asymptote, but it will never cross it.

Example: $y = \frac{x-2}{x-3}$

1ST No factoring is possible.
2ND No canceling is possible.
3RD $x - 3 = 0$ when $x = 3$.

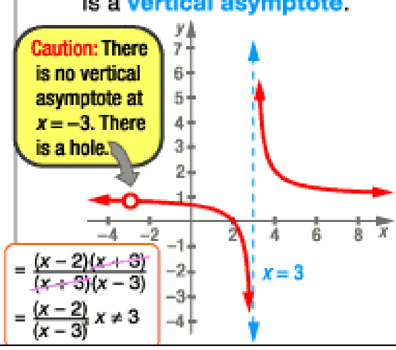
The line described by $x = 3$ is a **vertical asymptote**.



Example: $y = \frac{x^2 + x - 6}{x^2 - 9}$

1ST Factor.
2ND Cancel.
3RD $x - 3 = 0$ when $x = 3$.

The line described by $x = 3$ is a **vertical asymptote**.



For the rational function on the far left, no factoring or canceling is possible.

When you set the denominator equal to zero, you get an equation that you can solve to produce $x = 3$. This is the equation of the vertical asymptote for the function.

The function on the near side is more complicated. It can be factored and the numerator and denominator have a common factor. Canceling produces the same function as on the far left, except that it is not defined for $x = 3$. This produces a hole in the graph, not a vertical asymptote.

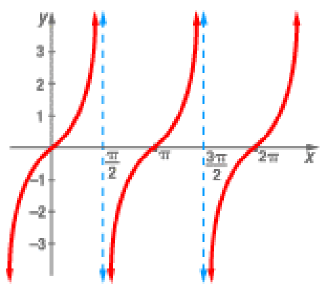
Horizontal Asymptotes and Infinite Limits

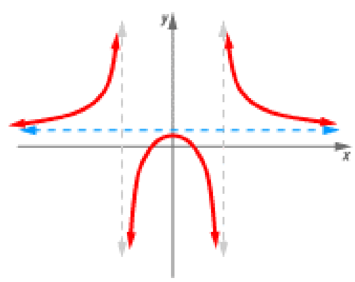
key concepts:

- **Asymptotes** are lines that the graph of a function approaches. A **horizontal asymptote** to the graph of a function f is a line whose equation is $y = a$ where $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a$.
- Identify horizontal asymptotes by taking the limit of the function as x approaches positive or negative infinity.
- Examine the highest-powered term in the numerator and the highest-powered term in the denominator when determining the limit of a rational function. The expression ∞/∞ is an **indeterminate form**.

Vertical asymptotes vs horizontal asymptotes

vertical asymptotes	horizontal asymptotes
Where is the function undefined such that its graph is shooting up or plunging down?	As the x-values get really large (or really small), does the function level off to some value?





A **horizontal asymptote** is present when the graph of a function levels off at positive infinity or negative infinity. Because it is a horizontal line, the equation will be of the form $y = a$.

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Limits at infinity

Consider $f(x) = \dfrac{1}{x}$.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

The graph of $f(x)$ has a **horizontal asymptote** at $y = 0$.

x	$\dfrac{1}{x}$
10	0.1
100	0.01
1,000	0.001
1,000,000	0.000001
1,000,000,000	0.000000001
↓	↓
∞	0

To understand how the function is behaving at infinity you need to take its limit at infinity. This will empower you to identify horizontal asymptotes.

To take a limit at infinity (∞), you need to recall that infinity represents a progression of increasing values. In this case, as x goes to ∞ (or gets larger and larger), $1/x$ goes to 0. Therefore the limit of the function at ∞ is 0, and there is a horizontal asymptote at $y = 0$.

Example: $\lim_{x\to\infty} x^3$

$$\lim_{x\to\infty} x^3 = \infty$$

This graph has no **horizontal asymptote**.

x	x^3
10	1000
100	1,000,000
↓	↓
∞	∞

This chart shows that as x gets increasingly large, x^3 gets increasingly large, too. Therefore the limit of the function at ∞ is ∞. There is no horizontal asymptote. The function never levels off away from the origin.

Horizontal Asymptotes and Infinite Limits

Exotic limits

Example: $\lim_{x\to\infty} \frac{3x}{2x^2+1}$

Q Who is approaching infinity faster?

$$\lim_{x\to\infty} \frac{3x}{2x^2+1} = 0$$

horizontal asymptote given by $y = 0$

Coefficients do not greatly influence the growth of the expression.

Higher powers approach infinity faster.

x	x^2
10	100
100	10,000
1000	1,000,000

A The denominator.

When evaluating the limit of a rational function at infinity, it is useful to ask the question "Which part is approaching infinity faster?"

Higher powers will approach infinity faster than lower powers. Identify the highest-powered term in the numerator and compare it with the highest-powered term in the denominator. In this case, the denominator approaches infinity faster, so the limit is 0. Therefore there is a horizontal asymptote at $y = 0$.

Example: $\lim_{x\to\infty} \frac{5x^3}{-2x^2+x-7}$

Q Who is approaching infinity faster?

$$\lim_{x\to\infty} \frac{5x^3}{-2x^2+x-7} = \lim_{x\to\infty} \frac{5x}{-2}$$
$$= -\infty$$

A The numerator.

no **horizontal asymptote**

Here the highest-powered term resides in the numerator. In the denominator, the x-term and the constant do not contribute much to the behavior of the function at infinity. You can actually ignore them and concentrate on the $5x^3$ and $-2x^2$ terms to find the limit. After simplifying the new fraction, you can see that the numerator is going to infinity, but the denominator is negative. Therefore the limit equals $-\infty$. There is no horizontal asymptote.

Finding horizontal asymptotes

Find the horizontal asymptotes for $f(x) = \frac{x^2+x-6}{3x^2-9}$

$$\lim_{x\to\infty} \frac{x^2+x-6}{3x^2-9} = \lim_{x\to\infty} \frac{\cancel{x^2}}{3\cancel{x^2}}$$ Consider the highest powers.

$$= \lim_{x\to\infty} \frac{1}{3}$$ Cancel.

$$= \frac{1}{3}$$ The limit exists.

The graph has a **horizontal asymptote** given by $y = \frac{1}{3}$.

To evaluate this limit you must consider the highest-powered term in the numerator and denominator. For the sake of the limit, you can ignore the other terms. Notice that when you cancel the common factors of x^2, the result is the limit of 1/3 as x approaches ∞. The limit of 1/3 is 1/3.

You can conclude the existence of a horizontal asymptote at $y = 1/3$.

- If the highest power on the top is greater than the highest power on the bottom, then the limit does not exist.
- If the highest power on the bottom is greater than the highest power on the top, then the limit is 0.
- If the highest power on the top is the same as the highest power on the bottom, then the limit exists and is not 0.

Here is a generalization of the results above. You do not need to memorize them in order to evaluate limits at infinity.

Unit: Curve Sketching **Module:** Asymptotes

Graphing Functions with Asymptotes

key concepts:

- Identify **vertical asymptotes** for a **rational function** by factoring the numerator and denominator, canceling where possible, and determining where the resulting denominator is zero. A vertical asymptote to the graph of a function f is a line whose equation is $x = a$ where $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

- Identify **horizontal asymptotes** by taking the limit of the function as x approaches positive or negative infinity. A horizontal asymptote to the graph of a function f is a line whose equation is $y = a$ where $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a$.

- The behavior of a function can change from one side of a vertical asymptote to the other.

Finding the asymptotes

Challenge: Sketch the graph of $f(x) = \dfrac{x-2}{x-3}$.

vertical asymptotes

Find where the denominator is zero.

$x - 3 = 0$ when $x = 3$.

vertical asymptote: $x = 3$

horizontal asymptotes

Take the limit as x approaches infinity.

$$\lim_{x\to\infty} \frac{x-2}{x-3} = \lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} \frac{1}{1} = 1$$

horizontal asymptote: $y = 1$

When sketching the graph of a **rational function**, you should first look for asymptotes.

Since this expression cannot be factored further and none of the factors cancel, set the denominator equal to 0. Solving for x indicates that the function has a **vertical asymptote** at $x = 3$.

Then take the limit of the function as x approaches ∞. For the sake of this limit, you can ignore the constants. Canceling produces 1 for the limit. Thus the function has a **horizontal asymptote** at $y = 1$.

Next, you can find the first and second derivatives and determine the behavior of the function.

The graph of the function

Summit: Sketch the graph of $f(x) = \dfrac{x-2}{x-3}$.

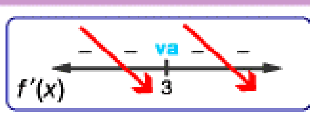
va: $x = 3$

ha: $y = 1$

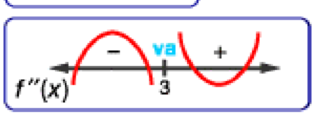
no **critical points**

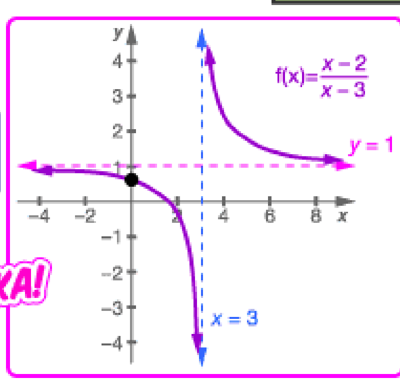
no **points of inflecion**

$$f'(x) = \frac{-1}{(x-3)^2}$$



$$f''(x) = \frac{2}{(x-3)^3}$$





EUREKA!

The first derivative of the function never equals 0. It is undefined at $x = 3$, but that is not a critical point because the function is not defined there. When you make your sign chart, mark $x = 3$ as the vertical asymptote.

The second derivative never equals 0 either. It is also undefined at $x = 3$, the vertical asymptote. Your sign chart should reflect this.

Although the concavity changes at $x = 3$, the function is not defined there. So there is no inflection point.

From the first derivative you can tell that the function is decreasing both to the left of the vertical asymptote and to its right. The second derivative indicates that the function is concave down on the left and concave up on the right.

Unit: Curve Sketching **Module:** Asymptotes

Functions with Asymptotes and Holes

key concepts:

- Identify **vertical asymptotes** for a **rational function** by: factoring the numerator and denominator, canceling where possible, and determining where the resulting denominator is zero. A vertical asymptote to the graph of a function f is a line whose equation is $x = a$ where $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.
- Identify **horizontal asymptotes** by taking the limit of the function as x approaches positive or negative infinity. A horizontal asymptote to the graph of a function f is a line whose equation is $y = a$ where $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a$.
- A **hole** (or **point discontinuity**) occurs in the graph of a function f at a point c if $\lim_{x\to c} f(x)$ exists and $f(c)$ is undefined or not equal to $\lim_{x\to c} f(x)$.

Finding the asymptotes

Sketch the graph of $f(x) = \dfrac{x^2+x-6}{x^2-9}$.

vertical asymptotes

Factor and cancel.

$$\frac{x^2+x-6}{x^2-9} = \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}(x-3)} \quad \text{Factor.}$$

$$= \frac{(x-2)}{(x-3)},\ x \neq -3 \quad \text{Cancel.}$$

The denominator is 0 when $x = 3$.

When graphing a rational function, first look for **vertical asymptotes**.

This function can be factored. The numerator and denominator have a common factor of $(x + 3)$, so cancel it. Make sure to promise not to evaluate the function at $x = -3$, because that would make the original expression undefined.

Now only $x = 3$ makes the denominator equal to zero, so that gives location of the vertical asymptote.

A hole

Sketch the graph of $f(x) = \dfrac{x^2+x-6}{x^2-9}$.

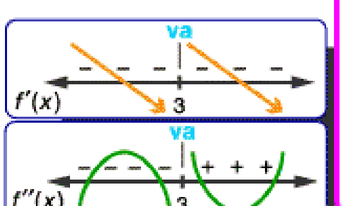
va: $x = 3$

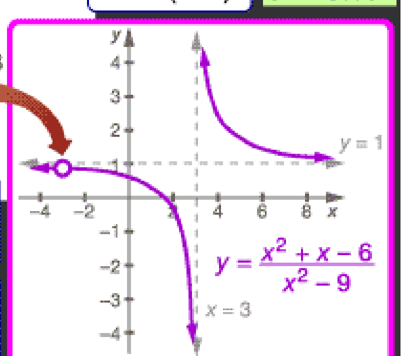
ha: $y = 1$

$f'(x) = \dfrac{-1}{(x-3)^2}$ no critical points

$f''(x) = \dfrac{2}{(x-3)^3}$ no points of inflection

$f(x) = \dfrac{x^2+x-6}{x^2-9}$ does not exist at $x = -3$.





Notice that the simplified expression for the function resembles a function you have already graphed. It has a **horizontal asymptote** at $y = 1$, but no critical point and no points of inflection.

The function is decreasing both to the left of $x = 3$ and to the right. On the left It is concave down, and on the right it is concave up.

Notice that there is a **hole** at $x = -3$. Since the function cannot be evaluated at this point, the graph skips over it, indicated by an open circle. Since the function is not continuous at that point, it is also called a **point discontinuity**.

Unit: Curve Sketching **Module:** Asymptotes

Functions with Asymptotes and Critical Points

key concepts:

- Identify **vertical asymptotes** for a **rational function** by: factoring the numerator and denominator, canceling where possible, and determining where the resulting denominator is zero. A vertical asymptote to the graph of a function f is a line whose equation is $x = a$ where $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

- Identify **horizontal asymptotes** by taking the limit of the function as x approaches positive or negative infinity. A horizontal asymptote to the graph of a function f is a line whose equation is $y = a$ where $\lim_{x \to \infty} f(x) = a$ or $\lim_{x \to -\infty} f(x) = a$.

- The behavior of a function can change from one side of a vertical asymptote to the other.

Finding the asymptotes

Challenge: Sketch the graph of $f(x) = \dfrac{x}{x^2 - 1}$.

vertical asymptotes

Factor.

$$f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x+1)(x-1)}$$

The denominator is 0 when $x = -1$ or $x = 1$.

vertical asymptotes: $x = -1$ or $x = 1$

horizontal asymptotes

Take the limit as x approaches infinity.

$$\lim_{x \to \infty} \frac{x}{x^2 - 1} = \lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x} = 0$$

horizontal asymptote: $y = 0$

When graphing a **rational function**, first look for **vertical asymptotes** and **horizontal asymptotes**.

The denominator of this function can be factored, but nothing cancels. There are two vertical asymptotes.

Taking the limit of the function at infinity indicates that there is a horizontal asymptote at $y = 0$. Notice that the degree of the denominator is greater than that of the numerator.

The graph of the function

Summit: Sketch the graph of $f(x) = \dfrac{x}{x^2 - 1}$.

va: $x = 1$, $x = -1$

ha: $y = 0$

$f'(x)$

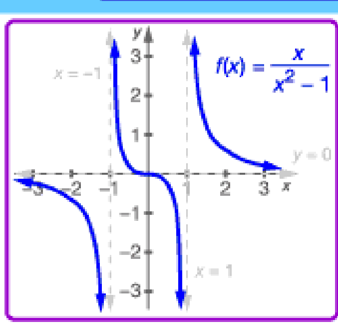
$f'(x) = \dfrac{-(x^2+1)}{(x^2-1)^2}$

no critical points

$f''(x)$

$f''(x) = \dfrac{(2x)(x^2+3)}{(x^2-1)^3}$

inflection point: $x = 0$



The first derivative is never equal to 0. It is only undefined at the locations of the vertical asymptotes, which are indicated on the sign chart.

The second derivative equals 0 at $x = 0$. According to the sign chart, the concavity changes at that point, so it is a point of inflection.

Use the asymptotes and sign charts to graph the function. The vertical asymptotes partition the plane into three regions, and the function is decreasing on each region. On the leftmost region the function is concave down. In the middle region the function changes from concave up to concave down at $x = 0$. On the rightmost region the function is concave up.