Calculus Lecture Notes

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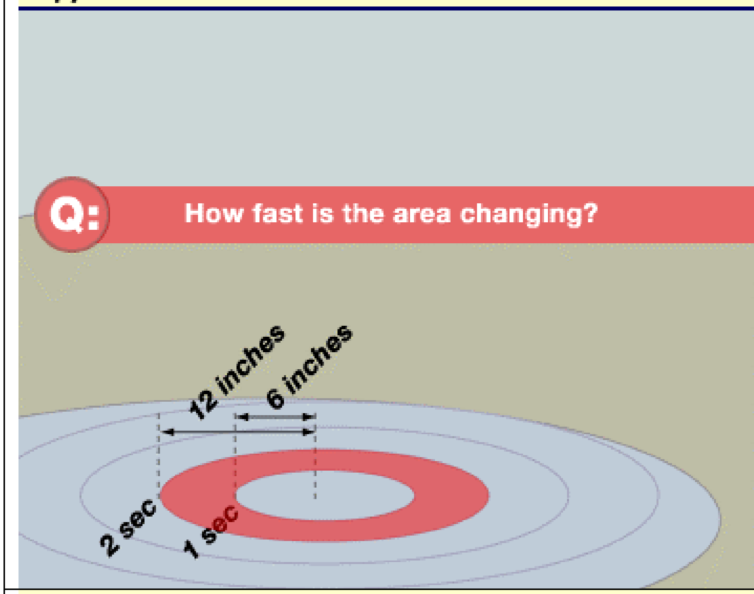
Unit: Practical Application of the Derivative **Module:** Related Rates

The Pebble Problem

key concepts:

- **Related rate** problems involve using a known rate of change to find an associated rate of change.
- The three steps to problem solving are understanding what you want, determining what you know, and finding a connection between the two.

Ripples



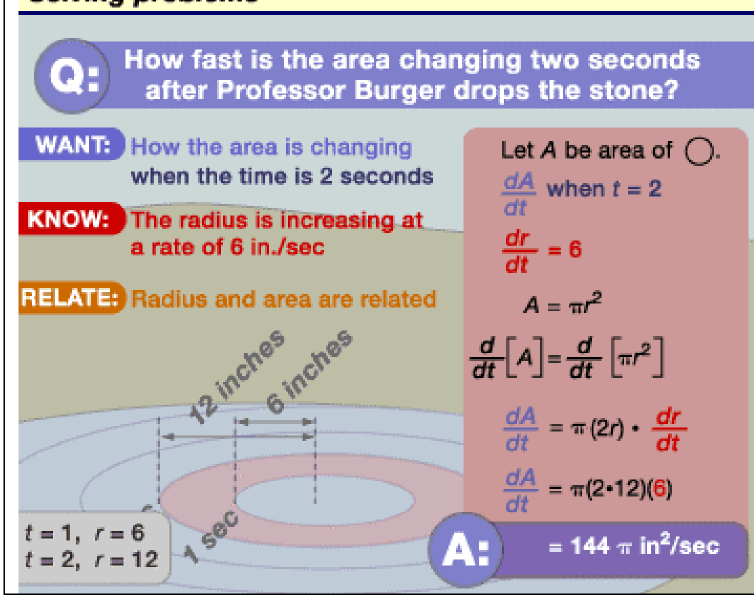
If you drop a stone into a body of water, ripples form across the surface of the water.

Suppose you are told that the radius of a ripple is increasing at a rate of 6 inches per second.

What is the rate of change in the area enclosed by the ripple?

This is an example of a **related rate**. Related rate questions ask you to find information about one rate given information about another.

Solving problems



You want to know the rate of change in the area enclosed by the ripple.

You are given the rate of change of the radius of the ripple. You also know the formula for the area of a circle.

Since area is expressed in terms of the radius, you can take the derivative of the area equation with respect to time. Notice that the derivative of the area equation includes two variables—the radius and the rate of change of the radius.

Substitute the values for the radius and the rate of change into the equation to find the rate of change in the area.

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The Ladder Problem

key concepts:

- **Related rate** problems involve using a known rate of change to find an associated rate of change.
- Use implicit differentiation when you cannot write the dependent variable in terms of the independent variable.

Studying a falling ladder

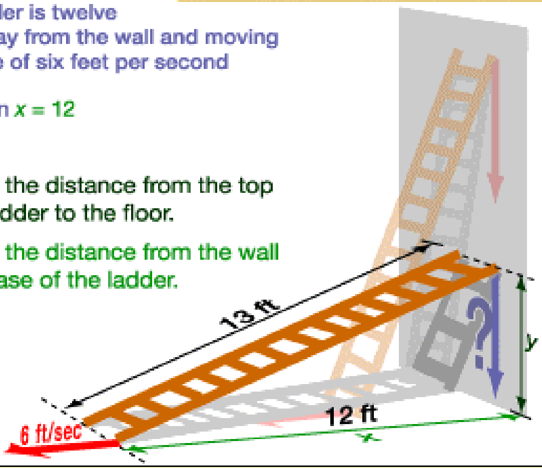
WANT: How fast the top of the ladder is falling at the very instant when the bottom of the ladder is twelve feet away from the wall and moving at a rate of six feet per second

Q: As the bottom of the ladder slides away from the wall, how fast does the top fall?

$\frac{dy}{dt}$ when $x = 12$

Let y be the distance from the top of the ladder to the floor.

Let x be the distance from the wall to the base of the ladder.



When a ladder slides down a wall, the rate at which it falls downward is not necessarily equal to the rate at which the base of the ladder moves away from the wall.

Suppose you are given a 13-foot ladder and you are told that the ladder is moving 6 feet per second away from the wall when the ladder is 12 feet away from the wall. What is the rate of change in the y-direction?

This is another example of a **related rate**. The rate at which the ladder falls downward depends on the rate at which the ladder moves away from the wall.

Studying a falling ladder

WANT: $\frac{dy}{dt}$ when $x = 12$

KNOW: $\frac{dx}{dt} = 6$ ft/sec

RELATE: $x^2 + y^2 = 13^2$

Q: As the bottom of the ladder slides away from the wall, how fast does the top fall?

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[13^2\right]$$ Use implicit differentiation.

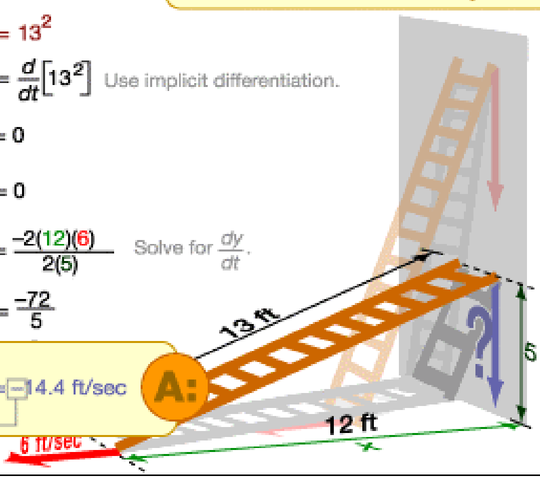
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(12)(6) + 2(5)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-2(12)(6)}{2(5)}$$ Solve for $\frac{dy}{dt}$.

$$= \frac{-72}{5}$$

A: $\frac{dy}{dt} = \boxed{-}14.4$ ft/sec (means rate is downward)



You want to know the rate of change in the y-direction when the ladder is 12 feet from the wall.

You are told that the rate of change in the x-direction when the ladder is 12 feet from the wall is 6 feet per second.

Since the Pythagorean theorem relates the distance away from the wall to the distance to the top of the ladder, their rates of change can be related by taking a derivative.

Notice that you can express the derivative of y with respect to time in terms of the derivative of x with respect to time, the x-value, and the y-value.

Substitute to find the value of dy/dt.

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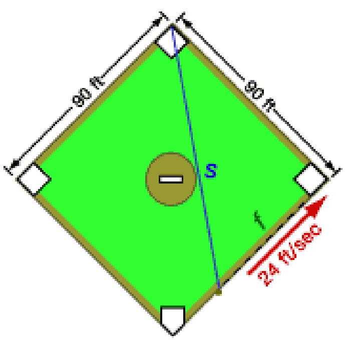
The Baseball Problem

key concepts:

- **Related rate** problems involve using a known rate of change to find an associated rate of change.
- Use implicit differentiation when you cannot write the dependent variable in terms of the independent variable.
- A negative value for the derivative means the original function is decreasing.

The baseball diamond

Q: How fast is the distance between second base and the batter changing at the instant that the batter is midway between home and first?



WANT: How fast the distance is changing at the very instant the batter is halfway between home and first

$\frac{ds}{dt}$ when $f = 45$ ft

Let s be the distance from the batter to second base.

Let f be the distance from the batter to first base.

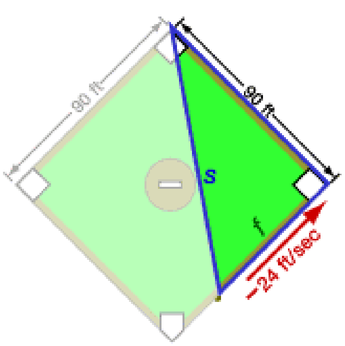
As a runner is moving towards first base, the distance between the runner and second base changes.

Suppose a runner is moving at 24 feet per second down the first baseline. How fast is the distance between the runner and second base changing at the instant the runner is halfway between home and first?

Since the distance between the runner and first base is shrinking, it makes sense to define the velocity that the runner is moving as a negative number instead of a positive one.

Using negative signs

Q: How fast is the distance between second base and the batter changing at the instant that the batter is midway between home and first?



WANT: $\frac{ds}{dt}$ when $f = 45$ ft

KNOW: $\frac{df}{dt} = -24$ ft/sec

RELATE: $f^2 + 90^2 = s^2$

$$\frac{d}{dt}\left[f^2 + 90^2\right] = \frac{d}{dt}\left[s^2\right]$$

$$2f\frac{df}{dt} + 0 = 2s\frac{ds}{dt}$$

$$2(45)(-24) = 2(45\sqrt{5})\frac{ds}{dt}$$

$$\cancel{90}(-24) = \cancel{90}\sqrt{5}\frac{ds}{dt}$$

$$-24 = \sqrt{5}\frac{ds}{dt}$$

$$\frac{ds}{dt} = -\frac{24}{\sqrt{5}} \text{ ft/sec}$$

Let s be the distance from the batter to second base.

Let f be the distance from the batter to first base.

The Pythagorean theorem gives you the distance between the runner and second base. Notice that the Pythagorean theorem relates three lengths together, one of which is a constant in this situation.

Use implicit differentiation to find the derivative of s with respect to time.

Substitute the known values into the equation and solve for ds/dt.

Notice that the rate is negative. This shows that the distance is decreasing as the runner moves.

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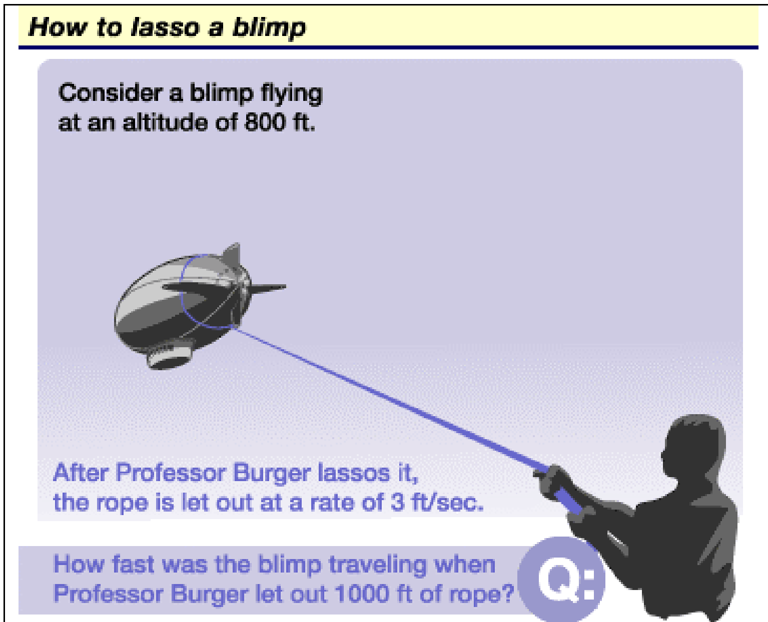
The Blimp Problem

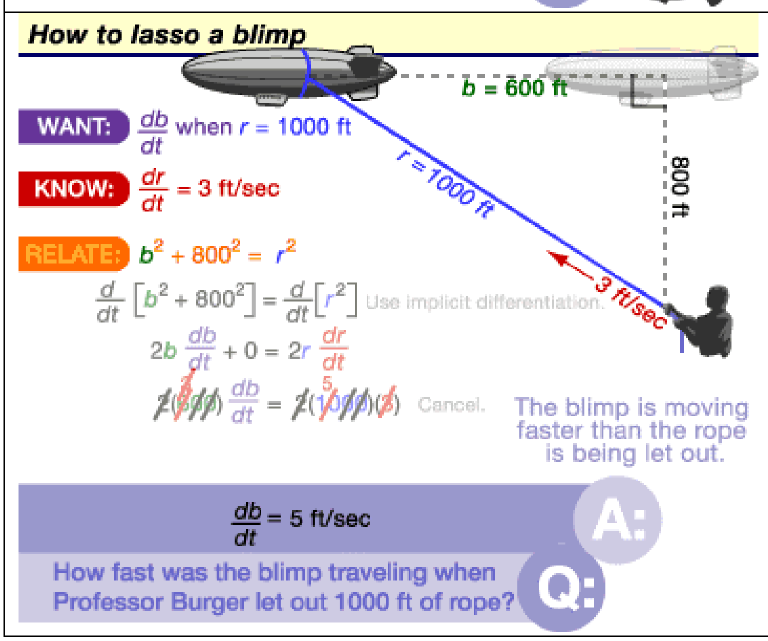
key concepts:

- **Related rate** problems involve using a known rate of change to find an associated rate of change.
- Use implicit differentiation when you cannot write the dependent variable in terms of the independent variable.



A blimp traveling overhead is tied to a rope. The rope is let out at a rate of 3 feet per second.

Assuming the blimp remains at a constant altitude of 800 feet, how fast is the blimp moving when 1000 feet of rope have been let out?



This situation can be modeled by a right triangle. Notice that it is not important to keep your diagram to scale. In fact, drawing to scale might mislead you, giving you false information.

The Pythagorean theorem relates the lengths of the sides of a right triangle to each other.

Use implicit differentiation to find the derivative of b with respect to time. Then substitute the known values into the equation.