Unit: Implicit Differentiation **Module:** Applying Implicit Differentiation

Using Implicit Differentiation

key concepts:

- Find the derivative of a relation by differentiating each side of its equation implicitly and solving for the derivative as an unknown. This process is called **implicit differentiation**.

Derivatives of relations

Consider $x^2 + y^2 = 1$.

$\frac{dy}{dx} = -\frac{x}{y}$

$\frac{dy}{dx} = -\frac{0}{1} = 0$

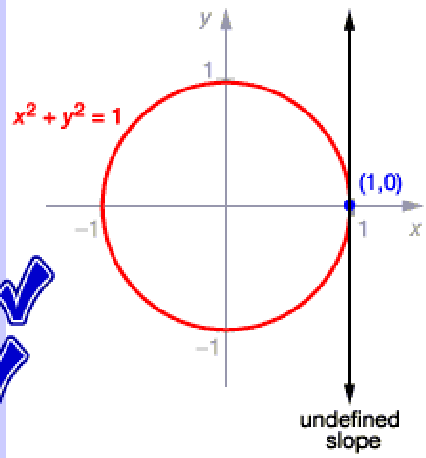
$\frac{dy}{dx} = -\frac{0}{-1} = 0$

$\frac{dy}{dx} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$

$\frac{dy}{dx} = -\frac{1/\sqrt{2}}{-1/\sqrt{2}} = -(-1) = 1$

$\frac{dy}{dx}$ is undefined

The derivative checks out.



Implicit differentiation often produces a derivative expressed in terms of more than one variable. When evaluating the slope of a line tangent to a point of a relation, it is necessary to substitute both the x-value and the y-value of the point into the derivative.

Notice that you could substitute any values for x and y into the derivative. However, only ordered pairs of the original relation produce reasonable answers.

Derivatives of exotic relations

Consider $x^2 + xy + y^2 = 3.75$. Find $\frac{dy}{dx}$.

not a function

Use implicit differentiation.

$$\frac{d}{dx}\left[x^2 + xy + y^2\right] = \frac{d}{dx}\left[3.75\right]$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[xy\right] + \frac{d}{dx}\left[y^2\right] = 0 \qquad \text{Differentiate each term.}$$

$$2x + x\frac{d}{dx}\left[y\right] + y \cdot 1 + \frac{d}{dx}\left[(y)^2\right] = 0 \qquad \text{Use the product rule.}$$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \qquad \text{Use the chain rule.}$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y) = -2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{-2x - y}{x + 2y}} \qquad \text{Solve for } \frac{dy}{dx}.$$

Here is a complicated-looking relation.

Find the derivative implicitly by taking the derivative of both sides of the implicit equation.

Now you can differentiate each term piece by piece.

Sometimes you will have to use different differentiation rules in the middle of a problem. Here the product and chain rules are both used.

Once you have differentiated each term you can solve for dy/dx.

Calculus Lecture Notes

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Unit: Implicit Differentiation **Module:** Applying Implicit Differentiation

Applying Implicit Differentiation

key concepts:

- Find the derivative of a relation by differentiating each side of its equation implicitly and solving for the derivative as an unknown. This process is called **implicit differentiation**.
- To find the equation of a line tangent to a curve, you need a point on the line and the slope of the line. To find the slope of the line you may need to substitute both the *x*-value and the *y*-value of the point into the derivative.

Relations and tangent lines

Consider $e^{xy} = y$.

Find the equation of the line tangent to this curve at the point (0,1).

To find the derivative of this relation you must use **implicit differentiation**.

Find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[e^{(xy)}\right] = \frac{d}{dx}[y]$$

Take the derivative of both sides of the equation.

$$e^{(xy)}\cdot\left(x\frac{dy}{dx} + y\right) = \frac{dy}{dx}$$

Notice that you must use the chain rule and the product rule to find the derivative.

$$xe^{xy}\frac{dy}{dx} + ye^{xy} = \frac{dy}{dx}$$

$$xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} = -ye^{xy}$$

$$(xe^{xy} - 1)\frac{dy}{dx} = -ye^{xy}$$

$$\frac{dy}{dx} = \frac{-ye^{xy}}{(xe^{xy} - 1)}$$

slope at (0,1)

$$\frac{dy}{dx} = \frac{-e^{0}}{0\cdot e^{(0,1)} - 1} = \frac{-1}{-1} = 1$$

Now substitute the point of tangency into the derivative to find the slope. Notice that you must substitute both the *x*-value and the *y*-value.

Use the point-slope formula.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 0)$$

$$\boxed{y = x + 1}$$

Use the point-slope formula to find the equation of the tangent line.

Fancy implicit differentiation

Find $\frac{dy}{dx}$ given $\sin(xy) + \ln(x + y) + (y^2 + 1)^3 = 1$.

This implicit equation will require several different differentiation rules to differentiate.

$$\frac{d}{dx}[\sin(xy)] + \frac{d}{dx}[\ln(x + y)] + \frac{d}{dx}\left[(y^2 + 1)^3\right] = 0$$

$$\cos(xy)\cdot\left(x\frac{dy}{dx} + y\right) + \frac{1}{x + y}\cdot\left(1 + \frac{dy}{dx}\right) + 3(y^2 + 1)^2\cdot\left(2y\frac{dy}{dx}\right) = 0$$

chain rule
product rule

$$\frac{d}{dx}[\sin(xy)] = \cos(xy)\cdot\left(x\frac{dy}{dx} + y\right)$$

chain rule

$$\frac{d}{dx}[\ln(x + y)] = \frac{1}{x + y}\cdot\left(1 + \frac{dy}{dx}\right)$$

chain rule
chain rule

$$\frac{d}{dx}\left[((y)^2 + 1)^3\right] = 3(y^2 + 1)^2\cdot\left(2y\frac{dy}{dx}\right)$$

It is a good idea to differentiate each term as a side-problem first and then to combine all of the results at the end of the problem.